

Interacting Hopf Algebras

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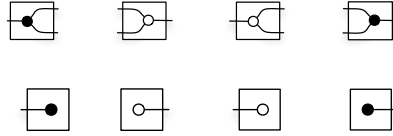
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Abstract. We introduce the symmetric monoidal theory of Interacting Hopf Algebras for a principal ideal domain R and we show that it characterizes the PROP of subspaces over the field of fractions k of R .

1 Introduction

Motivated by graphical formalisms for multi-qubit systems [5, 6] and concurrent systems [2, 3, 8, 16], in [1] we studied the theory of *interacting bialgebras*³ consisting of two monoid-comonoids structures, which we distinguish below graphically via white and black colouring.



The pairs monoid-comonoid of complementary colours form anti-separable bialgebras, while the pairs of the same colour form separable Frobenius algebras. The main result of [1] characterizes the free theory of interacting bialgebras \mathbb{IB} as the PROP \mathbb{SV} of \mathbb{Z}_2 -vector subspaces: the arrows $n \rightarrow m$ are vector subspaces of $\mathbb{Z}_2^n \times \mathbb{Z}_2^m$, with relational composition. In order to show this, we used Lack’s framework of distributive laws on PROPs [9]. The starting point was Lafont’s observation [10, Theorem 5] that the theory of anti-separable bialgebras \mathbb{AB} is precisely the PROP $\mathbf{Mat} \mathbb{Z}_2$ of \mathbb{Z}_2 -matrices. $\mathbf{Mat} \mathbb{Z}_2$ can be composed with its dual $\mathbf{Mat} \mathbb{Z}_2^{op}$ via a distributive law given by pullback: the result of this composition is $\mathbf{Span}(\mathbf{Mat} \mathbb{Z}_2)$, the PROP of spans in $\mathbf{Mat} \mathbb{Z}_2$. Dually, $\mathbf{Cospan}(\mathbf{Mat} \mathbb{Z}_2)$ arises from the distributive law of $\mathbf{Mat} \mathbb{Z}_2^{op}$ over $\mathbf{Mat} \mathbb{Z}_2$ given by pushout. The theories of $\mathbf{Span}(\mathbf{Mat} \mathbb{Z}_2)$ and $\mathbf{Cospan}(\mathbf{Mat} \mathbb{Z}_2)$ —called, respectively, \mathbb{IB}^{-w} and \mathbb{IB}^{-b} —are actually the same “up-to exchanging the colours”: they are the theory of \mathbb{IB} , but *without* the separability equation on precisely one of the white or black structures. The top and bottom faces in the cube below are pushout diagrams in the category of PROPs: the isomorphism between \mathbb{IB} and \mathbb{SV} then follows from the universal property of pushouts.

$$\begin{array}{ccccc}
 & & \mathbb{AB} + \mathbb{AB}^{op} & \xrightarrow{\quad} & \mathbb{IB}^{-w} \\
 & \swarrow & \cong \downarrow & \searrow & \downarrow \cong \\
 \mathbb{IB}^{-b} & \xrightarrow{\quad} & \mathbb{IB} & \xrightarrow{\quad} & \mathbb{IB} \\
 \cong \downarrow & & \downarrow & \downarrow & \downarrow \\
 \mathbf{Cospan}(\mathbf{Mat} \mathbb{Z}_2) & \xrightarrow{\quad} & \mathbf{Mat} \mathbb{Z}_2 + \mathbf{Mat} \mathbb{Z}_2^{op} & \xrightarrow{\quad} & \mathbf{Span}(\mathbf{Mat} \mathbb{Z}_2) \\
 & & \downarrow & \downarrow & \downarrow \\
 & & \mathbf{SV} & &
 \end{array}$$

In this work, we show that an analogous cube can be constructed by replacing \mathbb{Z}_2 with an arbitrary principal ideal domain R yielding the theory of *interacting Hopf algebras* \mathbb{IH}_R that char-

³ Also known as the phase free, undirected version of the ZX calculus [5, 6].

acterizes the PROP \mathbb{SV}_k of vector subspaces over the field of fractions k of R .

$$\begin{array}{ccccc}
 & \mathbb{H}\mathbb{A}_R + \mathbb{H}\mathbb{A}_R^{op} & \xrightarrow{\quad} & \mathbb{H}\mathbb{H}_R^w & \\
 & \cong \downarrow & & \downarrow \cong & \\
 \mathbb{H}\mathbb{H}_R^b & \xleftarrow{\quad} & \mathbb{H}\mathbb{H}_R & \xleftarrow{\quad} & \mathbb{H}\mathbb{H}_R^w \\
 \cong \downarrow & & \downarrow & \text{---} & \downarrow \\
 & \text{Mat } R + \text{Mat } R^{op} & \xrightarrow{\quad} & \text{Span}(\text{Mat } R) & \\
 \text{Cospan}(\text{Mat } R) & \xleftarrow{\quad} & \text{SV}_k & \xleftarrow{\quad} &
 \end{array} \quad (\boxplus)$$

We start by introducing the theory $\mathbb{H}\mathbb{A}_R$ which characterizes the PROP $\text{Mat } R$ of matrices over R (Section 3). A similar theory was given in [10] for matrices over fields. Beside adding generality, our approach makes the modular structure of $\mathbb{H}\mathbb{A}_R$ explicit by constructing it as (the quotient of) the composition of three different PROPs via distributive laws.

The next step is the introduction of $\mathbb{H}\mathbb{H}_R^w$ (Section 4, the w in the superscript stands for *white*) and the description of its compact closed structure (Section 5) which is instrumental in the proof that $\mathbb{H}\mathbb{H}_R^w$ is isomorphic to $\text{Span}(\text{Mat } R)$, the PROP of spans in $\text{Mat } R$ (Section 6). Since $\text{Span}(\text{Mat } R)$ is obtained as the composition of $\text{Mat } R^{op}$ and $\text{Mat } R$ via a distributive law given by pullback, by virtue of Lack’s framework [9], it is enough to check that the axiomatization of $\mathbb{H}\mathbb{H}_R^w$ equates all and only the pullbacks in $\text{Mat } R$. Soundness follows immediately, as each axiom in $\mathbb{H}\mathbb{H}_R^w$ corresponds to a pullback in $\text{Mat } R$. For completeness, we rely on the fact that every pullback in $\text{Mat } R$ can be constructed by computing the kernel of a certain matrix. Thus our proof reduces to showing how kernels can be computed inside the theory $\mathbb{H}\mathbb{H}_R^w$.

We then describe the rear faces of (\boxplus) (Section 7) and, in particular, we introduce the theory $\mathbb{H}\mathbb{H}_R^b$ —the b standing for *black*—which is simply the “photographic negative” of $\mathbb{H}\mathbb{H}_R^w$. The duality in $\text{Mat } R$ then guarantees that $\mathbb{H}\mathbb{H}_R^b$ is isomorphic to $\text{Cospan}(\text{Mat } R)$.

The theory of interacting Hopf algebras $\mathbb{H}\mathbb{H}_R$ is obtained by taking the union of the axioms of $\mathbb{H}\mathbb{H}_R^w$ and $\mathbb{H}\mathbb{H}_R^b$ (Section 8). By construction, the topmost face of (\boxplus) is a pushout in the category of PROPs, while the fact that also the bottom square is a pushout requires a formal proof (Section 9). Analogously to [1] the isomorphism between $\mathbb{H}\mathbb{H}_R$ and \mathbb{SV}_k follows from the universal property of pushouts. Finally, we paste together all the faces of the cube (Section 10) and give an explicit inductive definition of the induced isomorphism $S_{\mathbb{H}\mathbb{H}_R}: \mathbb{H}\mathbb{H}_R \rightarrow \mathbb{SV}_k$ that can be thought of as a semantics for terms of the theory $\mathbb{H}\mathbb{H}_R$. We conclude this report by illustrating some interesting instances of our result (Section 11).

2 Background

Composition of arrows $f: a \rightarrow b$, $g: b \rightarrow c$ is denoted by $f;g: a \rightarrow c$. $\mathbb{C}[a,b]$ indicates the set of arrows from a to b in a small category \mathbb{C} . For \mathbb{C} symmetric monoidal, we use notation \oplus for the monoidal product and $\sigma_{X,Y}: X \oplus Y \rightarrow Y \oplus X$ for the symmetry associated with $X, Y \in \mathbb{C}$. Given $\mathcal{F}: \mathbb{C}_1 \rightarrow \mathbb{C}_2$, we denote with $\mathcal{F}^{op}: \mathbb{C}_1^{op} \rightarrow \mathbb{C}_2^{op}$ the induced functor on the opposite categories of $\mathbb{C}_1, \mathbb{C}_2$. Given a category \mathbb{C} with pullbacks, its span bicategory has the objects of \mathbb{C} as 0-cells, spans of arrows of \mathbb{C} as 1-cells and span morphisms as 2-cells. We denote with $\text{Span}(\mathbb{C})$ the category obtained by identifying the isomorphic 1-cells and forgetting the 2-cells. Dually, if \mathbb{C} has pushouts we can form its bicategory of cospans and denote with $\text{Cospan}(\mathbb{C})$ the category obtained by identifying the isomorphic 1-cells and forgetting the 2-cells.

2.1 PROPs

A one sorted symmetric monoidal theory (SMT) is determined by a pair (Σ, E) where Σ is the signature: a set of elements $o: n \rightarrow m$ with *arity* n and *coarity* m . The set of Σ -terms is obtained by composing operations in Σ , the unit $id: 1 \rightarrow 1$ and the symmetry $\sigma_{1,1}: 2 \rightarrow 2$ with $;$ and \oplus . This is a purely formal process: given Σ -terms $t: k \rightarrow l$, $u: l \rightarrow m$, $v: m \rightarrow n$, we construct new Σ -terms $t;u: k \rightarrow m$ and $t \oplus v: k+n \rightarrow l+n$. The set E of equations contains pairs of Σ -terms

of the form $(t, t' : k \rightarrow l)$; here the only requirement is that t and t' have the same arity and coarity as Σ -terms.

One categorical approach used to study SMTs is the theory of PROPs [9, 11] (product and permutation categories). A PROP is a strict symmetric monoidal category with objects natural numbers, where \oplus on objects is addition. Morphisms between PROPs are strict symmetric monoidal functors that are identity on objects: PROPs and their morphisms form the category **PROP**. Now, given an SMT (Σ, E) , one (freely) obtains a PROP by letting the arrows $k \rightarrow l$ be the set of Σ -terms $k \rightarrow l$ taken modulo the laws of symmetric monoidal categories and the equations $t = t'$ for any $(t, t') \in E$. There is a natural graphical representation of these terms as arrows of monoidal categories (see [15]): we refer to these diagrams, which can be considered as Σ -terms modulo the laws of (strict) monoidal categories, as *circuits*. We will sometimes refer to PROPs which arise from SMTs as *syntactic* PROPs in order to distinguish such PROPs from *semantic* PROPs that are defined “directly”: for instance the PROP of functions \mathbb{F} where arrows $k \rightarrow l$ are simply functions $\{0, \dots, k-1\} \rightarrow \{0, \dots, l-1\}$.

PROPs can also be seen as living in a certain slice category. First, a PRO is simply a strict monoidal category with objects the natural numbers and tensor product on objects being addition. The morphisms of PROs are strict monoidal functors that are identity on objects. There is a PRO of particular interest: the PRO of permutations \mathbb{P} , where the homset $\mathbb{P}[k, l]$ is empty if $k \neq l$ and otherwise contains all the permutations on the set with k elements. PROPs can now be understood as objects of the slice category \mathbb{P}/\mathbf{PRO} , where **PRO** is the category of PROs and their morphisms. Morphisms of PROPs are thus simply morphisms of PROs that preserve the permutation structure. Working in the slice is also intuitive: e.g. \mathbb{P} is the initial PROP and in order to compute the coproduct $\mathbb{C} + \mathbb{D}$ in **PROP** one must identify the permutation structures in \mathbb{C} and \mathbb{D} .

2.2 Composing PROPs

In [9] Lack showed that the PROPs of co/commutative bialgebras and separable Frobenius algebras can be seen as arising from different ways of “composing” the PROPs of commutative monoids and cocommutative comonoids. Formally, this is understood in terms of distributive laws between monads. As shown in [17], the theory of monads can be developed in an arbitrary bicategory. Analogously to how small categories are monads in $\mathbf{Span}(\mathbf{Set})$, a PROP can be represented as a monad in a certain bicategory and any two PROPs \mathbb{T}_1 and \mathbb{T}_2 can be composed via a distributive law $\lambda : \mathbb{T}_2 ; \mathbb{T}_1 \rightarrow \mathbb{T}_1 ; \mathbb{T}_2$ between the associated monads. The monad $\mathbb{T}_1 ; \mathbb{T}_2$ yields a PROP whose arrows can be seen as *pairs* $(f, g) : n \rightarrow m$, where $f : n \rightarrow z$ is an arrow of \mathbb{T}_1 and $g : z \rightarrow m$ an arrow of \mathbb{T}_2 . A key observation for our purposes is that the graph of λ can be also seen as a set of (directed) equations of the form $(g, f) = (f', g')$. In fact, if \mathbb{T}_1 and \mathbb{T}_2 are syntactic PROPs then $\mathbb{T}_1 ; \mathbb{T}_2$ also has a presentation by operations and equations: this is the same as $\mathbb{T}_1 + \mathbb{T}_2$, plus the equations encoded by λ .

Beside [9], we refer the reader to Section 2 of [1] for a simple example of composing PROPs.

2.3 Computing the Kernel of Matrices over a PID

Throughout the paper we fix a principal ideal domain (PID) $R = \langle R, 0, 1, +, \cdot \rangle$. In this section we recall the basics of the theory of matrices with values in R , with particular attention to how their kernel (null space) can be computed. To this aim, we first recall a normal form for R -matrices called *Hermite Normal Form*.

Definition 1. *An $m \times n$ matrix A is said in Hermite Normal Form (HNF) if there is a natural number $r \leq n$ and a strictly increasing function $f : [r+1, n] \rightarrow [1, m]$ associating column i to a row $f(i)$, such that:*

1. *the first r columns of A have all entries with value 0;*
2. *for all columns i with $r+1 \leq i \leq n$, $A_{f(i), i} \neq 0$ and*

3. for all $j > f(i)$, $A_{j,i} = 0$.

There is an immediate observation stemming by the definition above.

Lemma 1. *Suppose that A is an $m \times n$ matrix in HNF and fix a column $i \leq n$. Then $A_{f(i),j} = 0$ for all columns $j < i$.*

Proof. If $j \leq r$ then $A_{f(i),j} = 0$ by property 1 of HNF. Otherwise, fix j such that $r < j < i$. Since f is strictly increasing, $f(i) > f(j)$. Then by property 3 of HNF, $A_{f(i),j} = 0$. \square

Hermite Normal Form generalizes Column Echelon Form to the setting of PIDs. Indeed, every R -matrix can be put into HNF by elementary column operations. We recall what those operations are:

Column Swap fixed $i, j \leq n$, replace column C_i with C_j and C_j with C_i ;

Column Sum replace column C_i with column $C_i + kC_j$, for some $k \in R$;

Unitary Multiplication replace column C_i with column uC_i , where $u \in R$ is a unitary element of the ring, i.e., it has a multiplicative inverse u^{-1} in R .

For later reference, we also mention elementary row operations, which are defined by replacing “column” with “row” in each of the three items above.

Proposition 1. *Every R -matrix A is column-equivalent to a unique matrix B in HNF.*

Proof. See e.g. [7, 12] \square

The transformation of A into B can be encoded as an invertible matrix U , obtained by applying to the identity matrix the sequence of elementary column operations allowing to pass from A to B . Then $B = AU$ and we can compute from U the kernel of A as follows.

Proposition 2. *For an $m \times n$ matrix A , let $B = AU$ be its HNF and $r \leq m$ the number of initial 0-columns of B . Then the first r columns of U form a basis for the kernel of A .*

Proof. A proof can be found for the PID of integers in [7, Prop. 2.4.9]. We reformulate the same argument here for an arbitrary PID R .

For $i \leq r$, let U_i be the i -th column of U . By definition $AU_i = B_i$, which is a 0-vector because $i \leq r$. Thus all first r columns of U are elements of the kernel of A . Conversely, let X be a vector such that $AX = 0$. Then $AX = AUU^{-1}X = BU^{-1}X$ because U is invertible. Let y_1, \dots, y_n be the entries of the vector $Y := U^{-1}X$. For each i in range $[r+1, n]$, we show that $y_i = 0$, by backward induction on i :

- if $i = n$, let $f(n)$ be given as in Definition 1. Since $BY = 0$, then the $f(n)$ -th entry of BY is

$$B_{f(n),1}y_1 + \dots + B_{f(n),n}y_n = 0. \quad (\triangle)$$

By Lemma 1, $B_{f(n),1}, \dots, B_{f(n),n-1}$ are all equal to 0, meaning by (\triangle) that $B_{f(n),n}y_n = 0$. By property 2 of HNF, $B_{f(n),n} \neq 0$ and thus, since R has no non-zero divisors, $y_n = 0$.

- For i with $r < i < n$, the $f(i)$ -th entry of BY is $B_{f(i),1}y_1 + \dots + B_{f(i),n}y_n = 0$ and by induction hypothesis $y_j = 0$ for all j such that $i < j \leq n$. By Lemma 1, $B_{f(i),1}, \dots, B_{f(i),i-1}$ are all equal to 0, which means, analogously to the base case, that $B_{f(i),i}y_i = 0$ and since $B_{f(i),i}$ then $y_i = 0$.

Thus we proved that the entries y_{r+1}, \dots, y_n of Y are equal to 0. Instead the first r entries of Y can be arbitrary, because the j -th row of BY , for $j \leq r$, is give by $B_{j,1}y_1 + \dots + B_{j,n}y_n = 0$ and we know that, by property 1 of HNF, the entries $B_{j,1}, \dots, B_{j,n}$ have value 0.

Therefore the kernel of B is generated by the first r canonical basis vectors C_1, \dots, C_r of R^n . Since $B = AU$, then UC_1, \dots, UC_r form a basis for the kernel of A . But those are just the first r columns of U : hence we have proven the statement of the theorem. \square

2.4 Categories of Matrices over a PID

In this section we set up a categorical environment for R -matrices. First, we fix notation for the following categories:

- the abelian category $\text{Mod } R$ of R -modules and linear maps;
- its full subcategory $\text{FMod } R$ consisting of the finitely-generated free R -modules and linear maps between them;
- the PROP $\text{Mat } R$ with arrows $n \rightarrow m$ being $m \times n$ R -matrices, where $;$ is matrix multiplication and \oplus is direct sum. The permutations are the rearrangements of the rows of the identity matrix.

There is an equivalence of categories between $\text{FMod } R$ and $\text{Mat } R$: a finitely-generated free R -module in $\text{FMod } R$, say of dimension n , is isomorphic to R^n and thus we can associate it with the object n in $\text{Mat } R$. A linear map $f: V \rightarrow W$ in $\text{FMod } R$ is represented by a matrix $M: n \rightarrow m$, where $V \cong R^n$ and $W \cong R^m$.

As $\text{FMod } R$ has biproducts given by direct sum, in $\text{Mat } R$ the object $n + m$ is the biproduct of n and m . Given two matrices $A: n \rightarrow z$ and $B: m \rightarrow z$ in $\text{Mat } R$, we denote with $(A|B): n + m \rightarrow z$ the matrix given by the universal property of $n + m$ as coproduct. Dually, given matrices $C: z \rightarrow n$ and $D: z \rightarrow m$, $(\begin{smallmatrix} C \\ D \end{smallmatrix}): z \rightarrow n + m$ is the matrix given by the universal property of $n + m$ as product. The graphical notation reflects the way in which these matrices are constructed, by putting A and B side-by-side and C above D .

For our purposes, it is of importance to discuss the existence of pullbacks and pushouts in $\text{Mat } R$. Two matrices $A: n \rightarrow z$ and $B: m \rightarrow z$ in $\text{Mat } R$ can be represented in $\text{Mod } R$ as arrows of type $R^n \rightarrow R^z$ and $R^m \rightarrow R^z$. Since $\text{Mod } R$ is an abelian category, their pullback may be formed by calculating the kernel $\text{Ker}(A|B): V \rightarrow R^n \oplus R^m$. Now, it is a well-known fact that (assuming the axiom of choice) R is a PID iff every submodule of a free R -module is itself free. The kernel object V is a submodule of $R^n \oplus R^m$ and therefore $V \cong R^r$ for some natural number $r \leq n + m$. We can then express the pullback of A and B in $\text{Mod } R$ as follows:

$$\begin{array}{ccccc}
 & & R^r & & \\
 & \swarrow C & \downarrow \text{Ker}(A|B) & \searrow D & \\
 R^n & & R^n \oplus R^m & & R^m \\
 & \searrow A & \downarrow (A|B) & \swarrow B & \\
 & & R^z & &
 \end{array}$$

Since $\text{Ker}(A|B)$ ranges over $R^n \oplus R^m$, it is of shape $(\begin{smallmatrix} C \\ D \end{smallmatrix}): R^r \rightarrow R^n \oplus R^m$ and postcomposition with the product projections $\pi_1: n \oplus m \rightarrow n$ and $\pi_2: n \oplus m \rightarrow m$ yields matrices C and D as in the diagram. It follows that we also have a pullback square in $\text{Mat } R$:

$$\begin{array}{ccc}
 & r & \\
 C \swarrow & & \searrow D \\
 n & & m \\
 A \searrow & & \swarrow B \\
 & z &
 \end{array}$$

Unfortunately, the same reasoning does not apply for pushouts. Given matrices $C: R^z \rightarrow R^n$ and $D: R^z \rightarrow R^m$, their pushout in $\text{Mod } R$ is formed by taking the cokernel $\text{Coker}(C|D): R^n \oplus R^m \rightarrow Q$. The object Q is not necessarily a free module, meaning that we cannot transfer the pushout diagram in $\text{Mat } R$. Nonetheless, $\text{Mat } R$ *does* have pushouts, because it is a self-dual category, with

isomorphism $\mathbf{Mat} R \cong \mathbf{Mat} R^{op}$ given by taking the transpose of a matrix. Therefore for purely formal reasons the pushout of matrices $C: z \rightarrow n$ and $D: z \rightarrow m$ in $\mathbf{Mat} R$ exists as the transpose of the pullback of transposed matrices $C^T: n \rightarrow z$ and $D^T: m \rightarrow z$ (note that this does not, in general, coincide with the pushout in $\mathbf{Mod} R$!).

2.5 From Matrices over a PID to Matrices over its Field of Fractions

Given the PID R , we fix notation k for its field of fractions. This is canonically constructed by letting elements of k be fractions $\frac{k_1}{k_2}$, where $k_1, k_2 \in R$, $k_2 \neq 0$ and $\frac{k_1}{k_2}$ represents an equivalence class of the relation $(k_1, k_2) \sim (k_3, k_4)$ on pairs of elements of R defined by

$$(k_1, k_2) \sim (k_3, k_4) \text{ if } k_1 \cdot k_4 = k_3 \cdot k_2.$$

As seen in Section 2.4, we can construct categories $\mathbf{Mod} k$, $\mathbf{FMod} k$ and $\mathbf{Mat} k$. Since k is a field, $\mathbf{FMod} k$ is just another name for the category of finite dimensional vector spaces over k : we know that any k -module is free and thus $\mathbf{Mod} k \cong \mathbf{FMod} k$. There is an obvious PROP morphism $I: \mathbf{Mat} R \rightarrow \mathbf{Mat} k$ interpreting a matrix with entries in R as a matrix with entries in k . Similarly, we have an inclusion $J: \mathbf{FMod} R \rightarrow \mathbf{FMod} k$. This yields the following commutative diagram, where \simeq denotes equivalence of categories.

$$\begin{array}{ccc} \mathbf{Mat} R & \xrightarrow{\simeq} & \mathbf{FMod} R \\ I \downarrow & & \downarrow J \\ \mathbf{Mat} k & \xrightarrow{\simeq} & \mathbf{FMod} k \end{array}$$

In the remaining of this section we record some facts that will be used in the developments of Section 9.

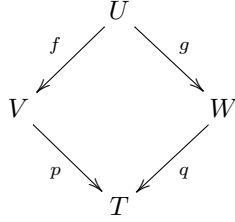
Lemma 2. $I: \mathbf{Mat} R \rightarrow \mathbf{Mat} k$ preserves pullbacks and pushouts.

Proof. Because the transpose operation induces a duality in both $\mathbf{Mat} R$ and $\mathbf{Mat} k$ (cf. Section 2.4), the morphism $\mathbf{Mat} R \rightarrow \mathbf{Mat} k$ preserves pullbacks iff it preserves pushouts. It is thus enough to show that it preserves pullbacks. This can be easily be proved directly as follows. Suppose that the diagram

$$\begin{array}{ccc} & r & \\ A \swarrow & & \searrow B \\ n & & m \\ C \searrow & & \swarrow D \\ & z & \end{array} \quad (\star)$$

is a pullback in $\mathbf{Mat} R$. We need to show that it is also a pullback in $\mathbf{Mat} k$. Suppose that, for some $P: q \rightarrow n$, $Q: q \rightarrow m$ in $\mathbf{Mat} k$ we have that $CP = DQ$ in $\mathbf{Mat} k$. Since R is a PID we can find least common multiples: thus let d be a common multiple of all the denominators that appear in P and Q . Then $dP: q \rightarrow n$, $dQ: q \rightarrow m$ are in $\mathbf{Mat} R$ and we have $C(dP) = d(CP) = d(DQ) = D(dQ)$. Since (\star) is a pullback in $\mathbf{Mat} R$, there exists a unique $H: q \rightarrow r$ with $AH = dP$ and $BH = dQ$. This means that we have found a mediating arrow, $H/d: q \rightarrow r$, in $\mathbf{Mat} k$ since $A(H/d) = AH/d = dP/d = P$ and similarly $B(H/d) = Q$. Uniqueness in $\mathbf{Mat} k$ can also be translated in a straightforward way to uniqueness in $\mathbf{Mat} R$. Basically if H' is another mediating morphism and d' is the least common multiple of denominators in H' then we must have $d'(H/d) = d'H'$ because of the universal property in $\mathbf{Mat} R$. Dividing both sides by d' yields the required equality. \square

Lemma 3. *Let the following be a pushout diagram in $\mathbf{FMod} \mathbf{k}$.*



Suppose that there exist $v \in V$, $w \in W$ such that $pv = qw$. Then there exists $u \in U$ with $fu = v$ and $gu = w$.

Proof. Pushouts in $\mathbf{FMod} \mathbf{k} \cong \mathbf{Mod} \mathbf{k}$ can be constructed by quotienting the vector space $V + W$ by the subspace generated by $\{(fu, gu) \mid u \in U\}$. Thus, if $p(v) = q(w)$ then there exists a chain u_1, u_2, \dots, u_k with $f(u_1) = v$, $g(u_1) = g(u_2)$, $f(u_2) = f(u_3)$, \dots , $f(u_{k-1}) = f(u_k)$ and $g(u_k) = w$. If $k = 1$ then we are finished. Otherwise, to construct an inductive argument we need to consider a chain u_1, u_2, u_3 with $f(u_1) = v$, $g(u_1) = g(u_2)$, $f(u_2) = f(u_3)$ and $g(u_3) = w$. Now $f(u_1 - u_2 + u_3) = f(u_1) - f(u_2) + f(u_3) = v$ and $g(u_1 - u_2 + u_3) = g(u_1) - g(u_2) + g(u_3) = w$, so we have reduced the size of the chain to one. \square

The proof of Lemma 3 relies on the fact that $\mathbf{Mod} \mathbf{k}$ reflects pushouts to $\mathbf{FMod} \mathbf{k}$, for \mathbf{k} a field. Observe that the same reasoning would not work for an arbitrary PID: as we saw in Section 2.4, pushouts in the category $\mathbf{Mod} \mathbf{R}$ are generally different from pushouts calculated in its subcategory $\mathbf{FMod} \mathbf{R}$.

3 The Theory of Matrices over a PID

In this section we construct the PROP $\mathbb{H}\mathbb{A}_{\mathbf{R}}$ of R -Hopf Algebras in steps, by composing together simpler algebraic theories. This modular reasoning is instrumental in showing that $\mathbb{H}\mathbb{A}_{\mathbf{R}}$ is a complete axiomatic presentation of $\mathbf{Mat} \mathbf{R}$.

Definition 2. *The PROP \mathbb{R} is freely generated by the signature consisting of a circuit \boxed{k} for each $k \in \mathbf{R}$ and the following equations, where k_1, k_2 range over \mathbf{R} .*

$$\boxed{1} = \text{parallel lines} \quad (\text{A1})$$

$$\boxed{k_1 \text{---} k_2} = \boxed{k_1 k_2} \quad (\text{A2})$$

We fix notation $\boxed{\blacktriangleright}$ for the circuit $\boxed{-1}$.

Definition 3. *The PROP \mathbb{M}^w of commutative monoids is freely generated by the signature consisting of circuits $\boxed{\circ}$, $\boxed{\circ}$ and the following equations.*

$$\boxed{\circ} = \text{parallel lines} \quad (\text{A3})$$

$$\boxed{\circ} = \text{parallel lines} \quad (\text{A4})$$

$$\boxed{\circ} = \text{parallel lines} \quad (\text{A5})$$

Definition 4. *The PROP \mathbb{C}^b of cocommutative comonoids is freely generated by the signature consisting of circuits $\boxed{\bullet}$, $\boxed{\bullet}$ and the following equations.*

$$\boxed{\bullet} = \text{parallel lines} \quad (\text{A6})$$

$$\boxed{\bullet} = \text{parallel lines} \quad (\text{A7})$$

$$\boxed{\bullet} = \text{parallel lines} \quad (\text{A8})$$

Lemma 4.

- There is a distributive law $\sigma : \mathbb{M}^w ; \mathbb{R} \Rightarrow \mathbb{R} ; \mathbb{M}^w$ yielding a PROP $\mathbb{R} ; \mathbb{M}^w$ presented by the equations of $\mathbb{R} + \mathbb{M}^w$ and, for all $k \in \mathbf{R}$:

$$\boxed{\begin{array}{c} k \\ \bullet \\ k \end{array}} = \boxed{\begin{array}{c} \bullet \\ \circ \end{array}} \quad (A9)$$

$$\boxed{\begin{array}{c} \circ \\ \bullet \end{array}} = \boxed{\begin{array}{c} \bullet \\ \circ \end{array}} \quad (A10)$$

- There is a distributive law $\tau: \mathbb{R}; \mathbb{C}^b \Rightarrow \mathbb{C}^b; \mathbb{R}$ yielding a PROP $\mathbb{C}^b; \mathbb{R}$ presented by the equations of $\mathbb{C}^b + \mathbb{R}$ and, for all $k \in \mathbb{R}$:

$$\boxed{\begin{array}{c} k \\ \bullet \end{array}} = \boxed{\begin{array}{c} \bullet \\ k \end{array}} \quad (A11)$$

$$\boxed{\begin{array}{c} k \\ \bullet \end{array}} = \boxed{\bullet} \quad (A12)$$

- There is a distributive law $\lambda: \mathbb{M}^w; \mathbb{C}^b \Rightarrow \mathbb{C}^b; \mathbb{M}^w$ yielding a PROP $\mathbb{C}^b; \mathbb{M}^w$ presented by the equations of $\mathbb{C}^b + \mathbb{M}^w$ and the following bialgebra equations:

$$\boxed{\begin{array}{c} \bullet \\ \circ \end{array}} = \boxed{\begin{array}{c} \bullet \\ \bullet \end{array}} \quad (A13)$$

$$\boxed{\begin{array}{c} \bullet \\ \bullet \end{array}} = \boxed{\begin{array}{c} \bullet \\ \bullet \end{array}} \quad (A15)$$

$$\boxed{\begin{array}{c} \circ \\ \bullet \end{array}} = \boxed{\begin{array}{c} \circ \\ \circ \end{array}} \quad (A14)$$

$$\boxed{\begin{array}{c} \circ \\ \bullet \end{array}} = id_0 \quad (A16)$$

Proof. For the first statement, let \mathbb{T} be the PROP freely generated by quotienting $\mathbb{R} + \mathbb{M}^w$ out of (A9) and (A10). Then \mathbb{R} and \mathbb{M}^w are subcategories of \mathbb{T} and equations (A9) and (A10) yield a representation of each circuit of \mathbb{T} as one of \mathbb{R} followed by one of \mathbb{M}^w , which is unique up-to-permutation. This forms a factorisation system in the sense of [9] and by [9, Th. 4.6] it induces a distributive law of PROPs as above. The second statement can be verified through an analogous reasoning. For the third statement, we refer to [9, §5.3]. \square

Proposition 3. *There is a distributive law $\theta: \mathbb{M}^w; (\mathbb{C}^b; \mathbb{R}) \Rightarrow (\mathbb{C}^b; \mathbb{R}); \mathbb{M}^w$ yielding a PROP $\mathbb{C}^b; \mathbb{R}; \mathbb{M}^w$ presented by the equations of $(\mathbb{R}; \mathbb{M}^w) + (\mathbb{C}^b; \mathbb{R}) + (\mathbb{C}^b; \mathbb{M}^w)$.*

Proof. In [4] it is proven that the natural transformation θ defined as $\lambda_{\mathbb{R}}; \mathbb{C}^b \sigma$ (or, equivalently, the natural transformation $\varphi := \mathbb{R} \lambda; \tau_{\mathbb{M}^w}: (\mathbb{R}; \mathbb{M}^w); \mathbb{C}^b \Rightarrow \mathbb{C}^b; (\mathbb{R}; \mathbb{M}^w)$) is a distributive law yielding the monad $\mathbb{C}^b; \mathbb{R}; \mathbb{M}^w$ if one can prove that the three distributive laws λ , σ and τ satisfy a compatibility condition called Yang-Baxter equation. This is given by commutativity of the following diagram, which can be easily verified by case analysis on the circuits of $\mathbb{M}^w; \mathbb{R}; \mathbb{C}^b$.

$$\begin{array}{ccccc} & & \mathbb{M}^w; \mathbb{C}^b; \mathbb{R} & \xrightarrow{\lambda_{\mathbb{R}}} & \mathbb{C}^b; \mathbb{M}^w; \mathbb{R} \\ & \nearrow \mathbb{M}^w \tau & & & \searrow \mathbb{C}^b \sigma \\ \mathbb{M}^w; \mathbb{R}; \mathbb{C}^b & & & & \mathbb{C}^b; \mathbb{R}; \mathbb{M}^w \\ & \searrow \sigma_{\mathbb{C}^b} & & & \nearrow \tau_{\mathbb{M}^w} \\ & & \mathbb{R}; \mathbb{M}^w; \mathbb{C}^b & \xrightarrow{\mathbb{R} \lambda} & \mathbb{R}; \mathbb{C}^b; \mathbb{M}^w \end{array}$$

As shown in [4], the multiplication for the monad $\mathbb{C}^b; \mathbb{R}; \mathbb{M}^w$ — and thus composition in the PROP $\mathbb{C}^b; \mathbb{R}; \mathbb{M}^w$ — is equivalently defined by θ or φ . This means that the equations holding in the PROP $\mathbb{C}^b; \mathbb{R}; \mathbb{M}^w$ are all those given by the distributive laws composing θ and φ , that is, λ , σ and τ . By the characterization of these three laws in Lemma 4, it follows that $\mathbb{C}^b; \mathbb{R}; \mathbb{M}^w$ can be presented as the sum of theories $(\mathbb{R}; \mathbb{M}^w) + (\mathbb{C}^b; \mathbb{R}) + (\mathbb{C}^b; \mathbb{M}^w)$. \square

Definition 5. *The PROP $\mathbb{H}\mathbb{A}_{\mathbb{R}}$ is defined as the quotient of $\mathbb{C}^b; \mathbb{R}; \mathbb{M}^w$ by the following equations, for all $k_1, k_2 \in \mathbb{R}$:*

$$\boxed{0} = \boxed{\bullet \circ} \quad (A17)$$

$$\boxed{\begin{array}{c} k_1 \\ \bullet \\ k_2 \end{array}} = \boxed{k_1 + k_2} \quad (A18)$$

Remark 1. The PROP $\mathbb{H}\mathbb{A}_R$ satisfies the axioms of Hopf Algebras. Indeed it inherits the bialgebra structure of $\mathbb{C}^b; \mathbb{M}^w$ and has as antipode the circuit $\boxed{\blacktriangleright}$, satisfying the following equations by virtue of (A1), (A18) and (A17):

$$\boxed{\bullet \blacktriangleright} = \boxed{\bullet \circ} = \boxed{\bullet \blacktriangleleft} \quad (\text{Hopf})$$

Remark 2. Since the PROP $\mathbb{H}\mathbb{A}_R$ is defined as a quotient of $\mathbb{C}^b; \mathbb{R}; \mathbb{M}^w$, it inherits the factorisation properties of composite PROPs. This means that any circuit $c \in \mathbb{H}\mathbb{A}_R[n, m]$ can be factorised as $s; r; t \in \mathbb{H}\mathbb{A}_R[n, m]$, where $s \in \mathbb{C}^b[n, z]$, $r \in \mathbb{R}[z, z]$ and $t \in \mathbb{M}^w[z, m]$ for some natural number z . Moreover, by (A18), we can assume that any port on the left boundary has at most one connection with any port on the right boundary, and by (A2),(A1) we know that any such connection passes through exactly one scalar circuit \boxed{k} . We say that a circuit $s; r; t \in \mathbb{H}\mathbb{A}_R[n, m]$ of this shape is in *matrix form*.

Circuits in matrix forms have an intuitive representation as R-matrices, as shown by the following example.

Example 1. Consider the circuit $t \in \mathbb{H}\mathbb{A}_R[3, 4]$ (on the right) and its representation as a 4×3 matrix (on the left).



$$M = \begin{pmatrix} k_1 & 0 & 0 \\ 1 & 0 & 0 \\ k_2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The values in M are calculated as follows. For each boundary of t , suppose a top-down enumeration of its ports, starting from 1. Then the entry $M_{i,j}$ has value $k \in R$ if, reading the circuit from the left to the right, one finds a path connecting the j^{th} port on the left to the i^{th} port on the right passing through a scalar circuit \boxed{k} , and 0 otherwise. By virtue of (A1), a connection not encountering any scalar circuit counts as one encountering $\boxed{1}$.

We will often represent with \boxed{A} the circuit, in matrix form, corresponding to the matrix A . We now make the matrix semantics of circuits in $\mathbb{H}\mathbb{A}_R$ formal.

Definition 6. The morphism $\mathcal{S}_{\mathbb{H}\mathbb{A}_R}: \mathbb{H}\mathbb{A}_R \rightarrow \text{Mat } R$ is defined inductively by

$$\begin{aligned} \boxed{\circ} &\mapsto ! & \boxed{\bullet} &\mapsto \mathfrak{j} & \boxed{\circ \bullet} &\mapsto (1 \ 1) & \boxed{\bullet \circ} &\mapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \boxed{k} &\mapsto (k) \\ s \oplus t &\mapsto \mathcal{S}_{\mathbb{A}\mathbb{B}}(s) \oplus \mathcal{S}_{\mathbb{A}\mathbb{B}}(t) & s; t &\mapsto \mathcal{S}_{\mathbb{A}\mathbb{B}}(s); \mathcal{S}_{\mathbb{A}\mathbb{B}}(t) \end{aligned}$$

where $!: 0 \rightarrow 1$ and $\mathfrak{j}: 1 \rightarrow 0$ are the arrows given by initiality and finality of 0 in $\text{Mat } R$. It can be checked that $\mathcal{S}_{\mathbb{H}\mathbb{A}_R}$ is well defined, as it respects the equations of $\mathbb{H}\mathbb{A}_R$.

Proposition 4. $\mathcal{S}_{\mathbb{H}\mathbb{A}_R}: \mathbb{H}\mathbb{A}_R \rightarrow \text{Mat } R$ is an isomorphism of PROPs.

Proof. Since the two categories have the same objects, it suffices to prove that $\mathcal{S}_{\mathbb{H}\mathbb{A}_R}$ is full and faithful. For this purpose, observe that, for a circuit c in matrix form, the matrix $\mathcal{S}_{\mathbb{A}\mathbb{B}}(c)$ can be computed as described in Example 1. Since any circuit is equivalent to one of this shape (cf. Remark 2), fullness and faithfulness follows by checking that the encoding of Example 1 is a 1-1 correspondence between matrices and circuits of $\mathbb{H}\mathbb{A}_R$ in matrix form. \square

The opposite category $\mathbb{H}\mathbb{A}_{\mathbb{R}}^{op}$ is also a PROP whose circuits we represent as those of $\mathbb{H}\mathbb{A}_{\mathbb{R}}$ “reflected about the y -axis” — the formal definition of this geometric transformation will be provided in Section 5. This means, for instance, that the circuit $\boxed{\bullet \curvearrowright} \in \mathbb{H}\mathbb{A}_{\mathbb{R}}[1, 2]$ has $\boxed{\curvearrowright \bullet} \in \mathbb{H}\mathbb{A}_{\mathbb{R}}^{op}[2, 1]$ as its contravariant counterpart. The PROP $\mathbb{H}\mathbb{A}_{\mathbb{R}}^{op}$ is isomorphic to $\mathbf{Mat} \mathbb{R}^{op}$ via $S_{\mathbb{H}\mathbb{A}_{\mathbb{R}}}^{op} : \mathbb{H}\mathbb{A}_{\mathbb{R}}^{op} \rightarrow \mathbf{Mat} \mathbb{R}^{op}$. We follow the convention of considering matrices in $\mathbf{Mat} \mathbb{R}^{op}[n, m]$ as matrices in $\mathbf{Mat} \mathbb{R}[m, n]$. This means that, since $\boxed{\bullet \curvearrowright}$ is mapped to $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathbf{Mat} \mathbb{R}[1, 2]$, then $\boxed{\curvearrowright \bullet}$ is mapped to $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathbf{Mat} \mathbb{R}^{op}[2, 1]$. In conclusion, one should intuitively follow the same procedure of Example 1 to compute the matrix of a circuit in $\mathbb{H}\mathbb{A}_{\mathbb{R}}^{op}$, but reading the circuit from right to left — meaning that columns are ports on the right boundary and rows are ports on the left boundary.

4 Composing R-Hopf Algebras: $\mathbb{H}\mathbb{H}_{\mathbb{R}}^w$

In this and the next sections we consider the task of giving an equational presentation of the theory obtained by composing $\mathbb{H}\mathbb{A}_{\mathbb{R}}$ and $\mathbb{H}\mathbb{A}_{\mathbb{R}}^{op}$. It will be provided by the PROP $\mathbb{H}\mathbb{H}_{\mathbb{R}}^w$ of “interacting R-Hopf Algebras”, where the superscript w represents the fact that there are axioms concerning the white structure, namely (S6)-(S9). While several corresponding properties of the black structure are derivable, the black counterparts of (S6) and (S7) do not, in general, hold in $\mathbb{H}\mathbb{H}_{\mathbb{R}}^w$.

Later on, in Section 7.2, we will introduce $\mathbb{H}\mathbb{H}_{\mathbb{R}}^b$, in which the axioms on the black and white structures are asymmetric in the opposite way. Roughly speaking, $\mathbb{H}\mathbb{H}_{\mathbb{R}}^b$ is the result of composing $\mathbb{H}\mathbb{A}_{\mathbb{R}}$ and $\mathbb{H}\mathbb{A}_{\mathbb{R}}^{op}$ in a different way.

Definition 7. The PROP $\mathbb{H}\mathbb{H}_{\mathbb{R}}^w$ is given by quotienting $\mathbb{H}\mathbb{A}_{\mathbb{R}} + \mathbb{H}\mathbb{A}_{\mathbb{R}}^{op}$ out of the following equations, where k, k_1, k_2 range over \mathbb{R} and $m = p_1 \cdot k_1 = p_2 \cdot k_2$ is the least common multiple of k_1 and k_2 .

$$\boxed{k_1 \curvearrowright k_2} = \boxed{\curvearrowright p_1 \curvearrowright p_2} \quad (\text{S1})$$

$$\boxed{\bullet \curvearrowright \bullet} = \boxed{\bullet \curvearrowright \bullet} = \boxed{\bullet \curvearrowright \bullet} \quad (\text{S2})$$

$$\boxed{\bullet \curvearrowright \bullet} = \boxed{\bullet \curvearrowright \bullet} = \boxed{\bullet \curvearrowright \bullet} \quad (\text{S3})$$

$$\boxed{\bullet \curvearrowright \bullet} = \boxed{\bullet \curvearrowright \bullet} \quad (\text{S4})$$

$$\boxed{\bullet \curvearrowright \bullet} = \boxed{\bullet \curvearrowright \bullet} \quad (\text{S5})$$

$$\boxed{k \curvearrowright \bullet} = \boxed{\bullet \curvearrowright} \quad (\text{S6})$$

$$\boxed{\bullet \curvearrowright} = id_0 \quad (\text{S7})$$

$$\boxed{k \curvearrowright \bullet} = \boxed{k \curvearrowright \bullet} \quad (\text{S8})$$

$$\boxed{\bullet \curvearrowright k} = \boxed{\bullet \curvearrowright k} \quad (\text{S9})$$

The following are some of the derived laws of $\mathbb{H}\mathbb{H}_{\mathbb{R}}^w$ (cf. Appendix C).

$$\boxed{\bullet \curvearrowright \bullet} = \boxed{\bullet \curvearrowright \bullet} \quad (\text{D1})$$

$$\boxed{\bullet \curvearrowright \bullet} = \boxed{\bullet \curvearrowright \bullet} \quad (\text{D2})$$

$$\boxed{\bullet \curvearrowright} = \boxed{\bullet \curvearrowright} \quad (\text{D3})$$

$$\boxed{\bullet \curvearrowright k} = \boxed{\bullet \curvearrowright k} \quad (\text{D4})$$

$$\boxed{k \curvearrowright \bullet} = \boxed{k \curvearrowright \bullet} \quad (\text{D5})$$

$$\boxed{\bullet \curvearrowright k} = \boxed{\bullet \curvearrowright k} \quad (\text{D6})$$

$$\boxed{k \curvearrowright \bullet} = \boxed{k \curvearrowright \bullet} = \boxed{k \curvearrowright \bullet} \quad (\text{D7})$$

Equation (D3) states that the antipodes of $\mathbb{H}\mathbb{A}_{\mathbb{R}}$ and $\mathbb{H}\mathbb{A}_{\mathbb{R}}^{op}$ coincide in $\mathbb{H}\mathbb{H}_{\mathbb{R}}^w$ and allows us to use the same notation $\boxed{\bullet \curvearrowright}$ for the two of them.

5 Compact Closed Structure

In this section we show that \mathbb{HH}_R^w is a compact closed category. This requires the following ingredients. Each object n of \mathbb{HH}_R^w is assigned a dual object n^* , which we set as n itself. Also, we associate n with circuits $\eta_n: 0 \rightarrow n + n$ and $\epsilon_n: n + n: 0$ defined by induction as follows:

$$\begin{aligned}
 \alpha_0: 2 \rightarrow 2 &:= \text{[Diagram: Two horizontal lines crossing each other]} & \alpha_{n+1}: 2(n+1) \rightarrow 2(n+1) &:= \text{[Diagram: A box labeled } \alpha_n \text{ with two input wires on the left and two output wires on the right, each with a dot]} \\
 \eta_0: 0 \rightarrow 0 &:= id_0 & \eta_{n+1}: 0 \rightarrow 2(n+1) &:= \text{[Diagram: A box labeled } \alpha_{n+1} \text{ with two input wires on the left and two output wires on the right, each with a dot]} \\
 \beta_0: 2 \rightarrow 2 &:= \text{[Diagram: Two horizontal lines crossing each other]} & \beta_{n+1}: 2(n+1) \rightarrow 2(n+1) &:= \text{[Diagram: A box labeled } \beta_n \text{ with two input wires on the left and two output wires on the right, each with a dot]} \\
 \epsilon_0: 0 \rightarrow 0 &:= id_0 & \epsilon_{n+1}: 2(n+1) \rightarrow 0 &:= \text{[Diagram: A box labeled } \beta_{n+1} \text{ with two input wires on the left and two output wires on the right, each with a dot]}
 \end{aligned}$$

For a more concrete grip on the definition above, we show the first values of η_n :

$$\eta_1 = \text{[Diagram: A box with two input wires on the left and two output wires on the right, each with a dot]} \quad \eta_2 = \text{[Diagram: A box with two input wires on the left and two output wires on the right, each with a dot]} \quad \eta_3 = \text{[Diagram: A box with two input wires on the left and two output wires on the right, each with a dot]}$$

For the sequel, we fix notation $\boxed{n \bullet}$ for η_n and $\boxed{\bullet n}$ for ϵ_n . Also, we let \boxed{n} be the circuit id_n . Similarly, $\boxed{n \circ}$ (respectively, $\boxed{\bullet n}$) denotes the tensor product of n times $\boxed{\circ}$ (respectively, $\boxed{\bullet}$).

Proposition 5. \mathbb{HH}_R^w is compact closed with structure given by $(\cdot)^*$, η_n and ϵ_n for each $n \in \mathbb{HH}_R^w$.

Proof. It suffices to verify the following equality, for each $n \in \mathbb{HH}_R^w$.

$$\boxed{n \bullet} \circ \boxed{n} = \boxed{n} \circ \boxed{\bullet n} = \boxed{n} \quad (\text{CC1})$$

The details of this derivation in \mathbb{HH}_R^w can be found in Appendix B. \square

As observed in [14, Remark 2.1], the operation $(\cdot)^*$ canonically extends to a contravariant functor defined on a circuit $c: n \rightarrow m$ by:

$$n \xrightarrow{\quad} \boxed{C} \xrightarrow{\quad} m \mapsto m \xrightarrow{\quad} \boxed{C^*} \xrightarrow{\quad} n := \text{[Diagram: A box labeled } C \text{ with input wires } n \text{ and } m \text{, and output wires } n \text{ and } m \text{]}$$

Corollary 1. For any circuit $c: n \rightarrow m$ of \mathbb{HH}_R^w ,

$$\begin{array}{ccc}
\begin{array}{c} n \\ \hline \boxed{C} \\ \hline m \end{array} \mapsto \begin{array}{c} n \\ \hline \boxed{C^*} \\ \hline m \end{array} & \text{(CC2)} & \begin{array}{c} m \\ \hline \boxed{C} \\ \hline n \end{array} \mapsto \begin{array}{c} m \\ \hline \boxed{C^*} \\ \hline n \end{array} \quad \text{(CC3)}
\end{array}$$

Proof. The following is the derivation of (CC2) in \mathbb{IH}_R^w . The one of (CC3) is analogous.

$$\begin{array}{c}
\begin{array}{c} n \\ \hline \boxed{C^*} \\ \hline m \end{array} \xrightarrow{\text{Def. } c^*} \begin{array}{c} n \\ \hline \boxed{C} \\ \hline m \end{array} \xrightarrow{\text{(CC1)}} \begin{array}{c} n \\ \hline \boxed{C} \\ \hline m \end{array}
\end{array}$$

□

We claim that one can really think of c^* as the circuit c “reflected about the y -axis”. To show this, we first give a precise definition of what such geometric transformation means. Given a circuit $c \in \mathbb{IH}_R^w[n, m]$, its reflection $c^R \in \mathbb{IH}_R^w[m, n]$ is defined by the contravariant functor $(\cdot)^R: \mathbb{IH}_R^w \rightarrow \mathbb{IH}_R^w$ given inductively as follows:

$$\begin{array}{ccc}
\begin{array}{c} \boxed{\bullet} \mapsto \boxed{\bullet} \\ \boxed{\circ} \mapsto \boxed{\circ} \end{array} & \begin{array}{c} \boxed{\bullet} \mapsto \boxed{\bullet} \\ \boxed{\circ} \mapsto \boxed{\circ} \end{array} & \begin{array}{c} \boxed{\circ} \mapsto \boxed{\circ} \\ \boxed{\bullet} \mapsto \boxed{\bullet} \end{array} \\
\begin{array}{c} \boxed{k} \mapsto \boxed{k} \\ \boxed{k} \mapsto \boxed{k} \end{array} & & \begin{array}{c} \boxed{k} \mapsto \boxed{k} \\ \boxed{k} \mapsto \boxed{k} \end{array} \\
\begin{array}{c} n \quad \boxed{C_1} \quad z \quad \boxed{C_2} \quad m \mapsto m \quad \boxed{C_2^R} \quad z \quad \boxed{C_1^R} \quad n \\ m \quad \boxed{C_1} \quad z' \quad \boxed{C_2} \quad m \mapsto z' \quad \boxed{C_1^R} \quad n \quad z \quad \boxed{C_2^R} \quad m \end{array}
\end{array}$$

Proposition 6. $c^* = c^R$ for all circuits $c: n \rightarrow m$ of \mathbb{IH}_R^w .

Proof. The proof is by induction on c . See Appendix B for the details of the various derivations. □

The reflection about the y -axis can be suitably restricted to a contravariant functor of type $\mathbb{HA}_R \rightarrow \mathbb{HA}_R^{op}$. Indeed, a circuit $c \in \mathbb{HA}_R[n, m]$ is also an element of $\mathbb{IH}_R^w[n, m]$ and the circuit $c^R = c^* \in \mathbb{IH}_R^w[m, n]$ is then an element of $\mathbb{HA}_R^{op}[m, n]$. It also follows by definition that, for $c \in \mathbb{HA}_R^{op}[n, m]$, its semantics $A = \mathcal{S}_{\mathbb{HA}_R}^{op}(c) \in \text{Mat } R^{op}[n, m]$ is equivalently given by $A = \mathcal{S}_{\mathbb{HA}_R}(c^*) \in \text{Mat } R[m, n]$, where the $n \times m$ matrix A is seen in the first case as an arrow of $\text{Mat } R^{op}$ and in the second case as one of $\text{Mat } R$.

6 Completeness of \mathbb{IH}_R^w

Recall from Section 2.4 that $\text{Mat } R$ has pullbacks and thus we can form the PROP $\text{Span}(\text{Mat } R)$. In this section we will develop the tools necessary to show the following characterization result.

Theorem 1. $\mathbb{IH}_R^w \cong \text{Span}(\text{Mat } R)$.

Our proof will essentially rely on the properties of composed PROPs. First, observe that one can form the PROP $\text{Span}(\mathbb{HA}_R) = \mathbb{HA}_R^{op}; \mathbb{HA}_R$ via a distributive law $\lambda_{pb}: \mathbb{HA}_R; \mathbb{HA}_R^{op} \rightarrow \mathbb{HA}_R^{op}; \mathbb{HA}_R$ which maps a cospan $(p, q) \in \mathbb{HA}_R; \mathbb{HA}_R^{op}$ into its pullback span $(f, g) \in \mathbb{HA}_R^{op}; \mathbb{HA}_R$. Since composition in $\text{Span}(\text{Mat } R)$ is also by pullback, by Proposition 4 we clearly have that $\text{Span}(\text{Mat } R) \cong \text{Span}(\mathbb{HA}_R)$.

Therefore, in order to show Theorem 1 it suffices to prove that all equations of \mathbb{IH}_R^w are derivable in $\text{Span}(\mathbb{HA}_R)$ (soundness) and viceversa (completeness).

For the soundness part, observe that the axioms of \mathbb{IH}_R^w are of two kinds. We have the axioms of $\mathbb{HA}_R + \mathbb{HA}_R^{op}$, which are also valid in $\text{Span}(\mathbb{HA}_R)$ by construction, and then we have axioms (S1)-(S9). It is immediate to check that they are all of the shape $p; q = f; g$, where p, g are circuits

of $\mathbb{H}\mathbb{A}_R$, q, f are circuits of $\mathbb{H}\mathbb{A}_R^{op}$ and (f^*, g) is the pullback of (p, q^*) in $\mathbb{H}\mathbb{A}_R$ (calculated in $\mathbf{Mat}\,R$). Since all the pullback squares of $\mathbb{H}\mathbb{A}_R$ yield a valid equation of $\mathbf{Span}(\mathbb{H}\mathbb{A}_R)$, it follows that axioms (S1)-(S9) are derivable in $\mathbf{Span}(\mathbb{H}\mathbb{A}_R)$ and thus we have the soundness statement.

It remains to show completeness. By construction, the valid equations of $\mathbf{Span}(\mathbb{H}\mathbb{A}_R)$ are all those of $\mathbb{H}\mathbb{A}_R + \mathbb{H}\mathbb{A}_R^{op}$ — which are also equations of $\mathbb{H}\mathbb{H}_R^w$ — and the ones given by pullback squares in $\mathbb{H}\mathbb{A}_R$. Thus we need to verify that any pullback in $\mathbb{H}\mathbb{A}_R$ (i.e., in $\mathbf{Mat}\,R$) yields an equation which is provable in $\mathbb{H}\mathbb{H}_R^w$.

Proposition 7. *For any pullback square in $\mathbf{Mat}\,R$ (as on the right), the corresponding circuit equation (on the left) is derivable in $\mathbb{H}\mathbb{H}_R^w$.*

$$\begin{array}{ccc}
 & C & r \\
 & \swarrow & \searrow \\
 n & & m \\
 & \swarrow & \searrow \\
 & A & B \\
 & \searrow & \swarrow \\
 & z &
 \end{array}
 \quad
 \begin{array}{c}
 \text{---} \boxed{A} \text{---} z \text{---} \boxed{B^*} \text{---} m \text{---} \\
 = \\
 \text{---} \boxed{C^*} \text{---} r \text{---} \boxed{D} \text{---} m \text{---}
 \end{array}$$

By the discussion above, the proof of Theorem 1 is completed by showing Proposition 7.

6.1 Circuits of Invertible Matrices

A key step towards a proof of Proposition 7 is to understand how to canonically represent pullback spans as circuits. As mentioned in Section 2.4, pullbacks in $\mathbf{Mat}\,R$ are given by kernels of matrices. Thus the question reduces to expressing the kernel computation in terms of circuits. In Section 2.3 we illustrated how an essential role in such a process is played by elementary column operations, which are encoded by invertible matrices. For this reason, we now prove some basic properties of the circuit representation of invertible matrices, which will be useful later for computing kernels in a circuit setting.

Lemma 5. *For $U \in \mathbf{Mat}\,R[n, n]$ invertible, the following holds in $\mathbb{H}\mathbb{H}_R^w$:*

$$\text{---} \boxed{U^{-1}} \text{---} n \text{---} = \text{---} \boxed{U^*} \text{---} n \text{---} \quad (1)$$

Proof. Recall that an invertible $n \times n$ R -matrix is one obtainable from the identity $n \times n$ matrix by application of elementary row operations. Thus we can prove our statement by induction on the number of applied operations.

The base case is the one in which no row operation is applied and thus $U = id_n$. Then we have the following equality in $\mathbb{H}\mathbb{H}_R^w$, yielding (1).

$$\text{---} \boxed{U^{-1}} \text{---} n \text{---} = \text{---} n \text{---} = \text{---} \boxed{U^*} \text{---} n \text{---}$$

Inductively, suppose that U is obtained by swapping two rows of an invertible matrix V . We can assume without loss of generality that the two rows are one immediately above the other, with j the number of rows above them and m the number of rows below, where $n = j + 2 + m$. In circuit terms, this means that

$$\text{---} \boxed{U} \text{---} n \text{---} = \text{---} \boxed{V} \text{---} \begin{array}{c} j \\ \text{---} \text{---} \text{---} \\ m \end{array} \text{---}$$

In order to show (1), it suffices to prove that the circuit representing U^* is the inverse of U , that is, $U; U^* = id_n = U^*; U$. This is given by the following derivations.

$$\begin{array}{c}
 \text{---} \boxed{V} \text{---} \begin{array}{c} j \\ \text{---} \text{---} \text{---} \\ m \end{array} \text{---} \boxed{V^*} \text{---} n \text{---} \xrightarrow{\text{Axiom SMCs}} \text{---} \boxed{V} \text{---} n \text{---} \boxed{V^*} \text{---} n \text{---} \xrightarrow{\text{IH}} \text{---} n \text{---} \\
 \text{---} \begin{array}{c} j \\ \text{---} \text{---} \text{---} \\ m \end{array} \boxed{V^*} \text{---} n \text{---} \boxed{V} \text{---} \begin{array}{c} j \\ \text{---} \text{---} \text{---} \\ m \end{array} \xrightarrow{\text{IH}} \text{---} \begin{array}{c} j \\ \text{---} \text{---} \text{---} \\ m \end{array} \text{---} \begin{array}{c} j \\ \text{---} \text{---} \text{---} \\ m \end{array} \xrightarrow{\text{Axiom SMCs}} \text{---} n \text{---}
 \end{array}$$

The next inductive case that we consider is the one of row sum. As above, we may assume that such operation is applied to adjacent rows of an invertible matrix V . The circuit representing U has the following shape:

$$n \text{---} \boxed{U} \text{---} n = n \text{---} \boxed{V} \text{---} \begin{array}{c} j \\ \text{---} \boxed{k} \text{---} \\ m \end{array}$$

and the following two derivations prove that U^\star is the inverse of U :

$$\begin{array}{c} n \text{---} \boxed{V} \text{---} \begin{array}{c} j \\ \text{---} \boxed{k} \text{---} \\ m \end{array} \text{---} \boxed{V^\star} \text{---} n \quad \stackrel{(D7)}{=} \quad n \text{---} \boxed{V} \text{---} n \text{---} \boxed{V^\star} \text{---} n \quad \stackrel{\text{IH}}{=} \quad n \text{---} \\ \\ \begin{array}{c} j \\ \text{---} \boxed{k} \text{---} \\ m \end{array} \text{---} \boxed{V^\star} \text{---} n \text{---} \boxed{V} \text{---} \begin{array}{c} j \\ \text{---} \boxed{k} \text{---} \\ m \end{array} \quad \stackrel{\text{IH}}{=} \quad \begin{array}{c} j \\ \text{---} \boxed{k} \text{---} \\ m \end{array} \text{---} \boxed{k} \text{---} \boxed{k} \text{---} \begin{array}{c} j \\ \text{---} \boxed{k} \text{---} \\ m \end{array} \quad \stackrel{(D7)}{=} \quad n \text{---} \end{array}$$

Finally, we have the inductive case in which U is obtained by V via multiplication of a row by a unitary element $i \in \mathbb{R}$. We denote with $i^{-1} \in \mathbb{R}$ the multiplicative inverse of i . The circuit representing U has the following shape, where $z + 1 + m = n$:

$$n \text{---} \boxed{U} \text{---} n = n \text{---} \boxed{V} \text{---} \begin{array}{c} z \\ \text{---} \boxed{i} \text{---} \\ m \end{array}$$

and we can derive the desired equalities in \mathbb{IH}_R^w as follows.

$$\begin{array}{c} n \text{---} \boxed{V} \text{---} \begin{array}{c} z \\ \text{---} \boxed{i} \text{---} \\ m \end{array} \text{---} \boxed{V^\star} \text{---} n \quad \stackrel{(S1)}{=} \quad n \text{---} \boxed{V} \text{---} n \text{---} \boxed{V^\star} \text{---} n \quad \stackrel{\text{IH}}{=} \quad n \text{---} \\ \\ \begin{array}{c} z \\ \text{---} \boxed{i} \text{---} \\ m \end{array} \text{---} \boxed{V^\star} \text{---} n \text{---} \boxed{V} \text{---} \begin{array}{c} z \\ \text{---} \boxed{i} \text{---} \\ m \end{array} \quad \stackrel{\text{IH}}{=} \quad \begin{array}{c} z \\ \text{---} \boxed{i} \text{---} \\ m \end{array} \quad \stackrel{(S1)}{=} \quad \begin{array}{c} z \\ \text{---} \boxed{i^{-1}} \text{---} \boxed{i^{-1}} \text{---} \boxed{i} \text{---} \boxed{i} \text{---} \boxed{i^{-1}} \text{---} \boxed{i^{-1}} \text{---} \\ m \end{array} \\ \\ \stackrel{(A2)}{=} \quad \begin{array}{c} z \\ \text{---} \boxed{i^{-1}} \text{---} \boxed{i^{-1}} \text{---} \\ m \end{array} \quad \stackrel{(S1)}{=} \quad n \text{---} \end{array}$$

□

For $U \in \text{Mat } \mathbb{R}[n, n]$ invertible and $r \leq n$, we define the r -restriction of U as the matrix $U_{\upharpoonright r} \in \text{Mat } \mathbb{R}[r, n]$ consisting of the first r columns of U .

Lemma 6. *Let $U \in \text{Mat } \mathbb{R}[n, n]$ be invertible and fix $r \leq n$. Then the following holds in \mathbb{IH}_R^w :*

$$r \text{---} \boxed{U_{\upharpoonright r}} \text{---} n = \begin{array}{c} r \\ \text{---} \boxed{U} \text{---} \\ n-r \end{array} n \quad (2)$$

Proof. Since U is invertible, it can be obtained by applying elementary row operations to the identity $n \times n$ matrix. The proof goes by induction on the number of such operations necessary to obtain U . In the base case, $U = id_n$ is obtained by applying no operations. The circuit representation of $id_{n \upharpoonright r}$ is given by removing the connections between the bottommost $n - r$ ports on the

right boundary to ports on the left boundary. Thus, by definition, we have the following equality, yielding (2).

$$\frac{r}{\text{Id}_n \upharpoonright r} \frac{n}{\text{Id}_n \upharpoonright r} = \frac{r}{\text{Id}_n \upharpoonright r}$$

The inductive case applies for U obtained by applying an elementary row operation to an invertible matrix V . Here we display the derivation for the case in which the last applied operation is row swap.

$$\frac{r}{U \upharpoonright r} \frac{n}{U \upharpoonright r} = \frac{r}{V \upharpoonright r} \frac{j}{m} \stackrel{\text{IH}}{=} \frac{r}{\text{Id}_n \upharpoonright r} \frac{j}{m} = \frac{r}{\text{Id}_n \upharpoonright r} \frac{j}{m}$$

Observe that row operations are given by *postcomposing* a circuit with V , whereas the restriction to the first r columns corresponds to *precomposing* a circuit with V . Thus these two transformations do not really interact one with the other and the derivation of (2) just consists in applying the inductive hypothesis. The remaining inductive cases are treated analogously. \square

Finally, we record a lemma concerning span isomorphisms. This directly concerns invertible \mathbf{R} -matrices, as they are precisely the isomorphisms in $\mathbf{Mat} \mathbf{R}$. Recall that objects of $\mathbf{Span}(\mathbf{Mat} \mathbf{R})$ are isomorphic classes of spans in $\mathbf{Mat} \mathbf{R}$: we identify $n \xleftarrow{A} z \xrightarrow{B} m$ and $n \xleftarrow{C} z \xrightarrow{D} m$ if there is an invertible matrix $U \in \mathbf{Mat} \mathbf{R}[z, z]$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & z & & \\ & A \nearrow & \uparrow U & \nwarrow B & \\ n & \xleftarrow{C} & z & \xrightarrow{D} & m \end{array} \quad (3)$$

The next statement guarantees that spans which are identified in $\mathbf{Span}(\mathbf{Mat} \mathbf{R})$ are not distinguished by the equational theory of $\mathbb{H}_{\mathbf{R}}^w$.

Lemma 7. *Let A, B, C, D, U be as in diagram (3). Then the following equation holds in $\mathbb{H}_{\mathbf{R}}^w$:*

$$\frac{n}{A^*} \frac{z}{B} \frac{m}{\text{Id}_m} = \frac{n}{C^*} \frac{z}{D} \frac{m}{\text{Id}_m}$$

Proof. Since $\mathbb{H}_{\mathbf{A}_{\mathbf{R}}} \cong \mathbf{Mat} \mathbf{R}$, commutativity of (3) yields the following equalities of circuits in $\mathbb{H}_{\mathbf{A}_{\mathbf{R}}}$:

$$\frac{z}{U} \frac{z}{U^{-1}} \frac{z}{\text{Id}_z} = \frac{z}{\text{Id}_z} = \frac{z}{U^{-1}} \frac{z}{U} \frac{z}{\text{Id}_z} \quad (4)$$

$$\frac{z}{C} \frac{n}{\text{Id}_n} = \frac{z}{U} \frac{z}{A} \frac{n}{\text{Id}_n} \quad (5) \quad \frac{z}{U^{-1}} \frac{z}{D} \frac{m}{\text{Id}_m} = \frac{z}{B} \frac{m}{\text{Id}_m} \quad (6)$$

Since $\mathbb{H}_{\mathbf{A}_{\mathbf{R}}}$ is a sub-theory of $\mathbb{H}_{\mathbf{R}}^w$, these equations are also valid in $\mathbb{H}_{\mathbf{R}}^w$. The statement of the lemma is then given by the following derivation.

$$\begin{aligned} \frac{n}{C^*} \frac{z}{D} \frac{m}{\text{Id}_m} &\stackrel{(4)}{=} \frac{n}{C^*} \frac{z}{U} \frac{z}{U^{-1}} \frac{z}{D} \frac{m}{\text{Id}_m} \\ &\stackrel{(6)}{=} \frac{n}{C^*} \frac{z}{U} \frac{z}{B} \frac{m}{\text{Id}_m} \\ &\stackrel{(5)}{=} \frac{n}{(U \frac{z}{A})^*} \frac{z}{U} \frac{z}{B} \frac{m}{\text{Id}_m} \\ &\stackrel{\text{Def. } (\cdot)^*}{=} \frac{n}{A^*} \frac{z}{U^*} \frac{z}{U} \frac{z}{B} \frac{m}{\text{Id}_m} \\ &\stackrel{\text{Lemma 5}}{=} \frac{n}{A^*} \frac{z}{U^{-1}} \frac{z}{U} \frac{z}{B} \frac{m}{\text{Id}_m} \\ &\stackrel{(4)}{=} \frac{n}{A^*} \frac{z}{B} \frac{m}{\text{Id}_m} \end{aligned}$$

\square

The importance of Lemma 7 may be better understood in view of Proposition 7. There we need to show that the circuit representing a cospan is equal to the one representing its pullback span. However, pullbacks are unique only up to isomorphisms. Lemma 7 guarantees that we can prove our statement w.l.o.g. for a canonical choice of the pullback span and the corresponding circuit.

6.2 Computing Kernels in \mathbb{HH}_R^w

In this section we describe how the kernel computation of a matrix can be formulated in terms of circuits, by only using transformations which are sound with respect to the equational theory of \mathbb{HH}_R^w .

Lemma 8. *Let B be an $m \times n$ R -matrix in HNF and r the number of initial 0-columns of B given by property 1 in Definition 1. Then the following holds in \mathbb{HH}_R^w :*

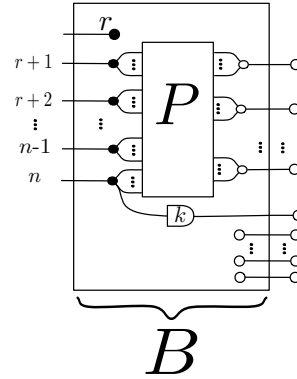
$$\text{---}_n \boxed{B} \text{---}_m \circ = \frac{r}{n-r} \bullet \circ$$

Proof. Essentially, what we have to show is that the kernel computation described in the proof of Proposition 2, when translated in terms of circuits, only uses valid equations of \mathbb{HH}_R^w . Since B is in HNF, the corresponding circuit can be assumed of a particular shape, that we depict on the right.

Here P is some circuit only made of symmetries $\boxed{\text{---}} \boxed{\text{---}}$ and

scalars \boxed{k} as basic components. By property 1 of HNF,

the first r columns of B only have 0 entries, meaning that the topmost r ports on the left boundary are not connected to the right boundary. Also, by Lemma 1 we know that the $f(n)$ -th row of B (where $f: [r+1, n] \rightarrow [1, m]$ is as in Definition 1) has only one non-0 value $k \in R$, at position $B_{f(n),n}$. In circuit terms, this allows us to assume that the $f(n)$ -th port on the right boundary only connects to the n -th and last port on the left boundary.

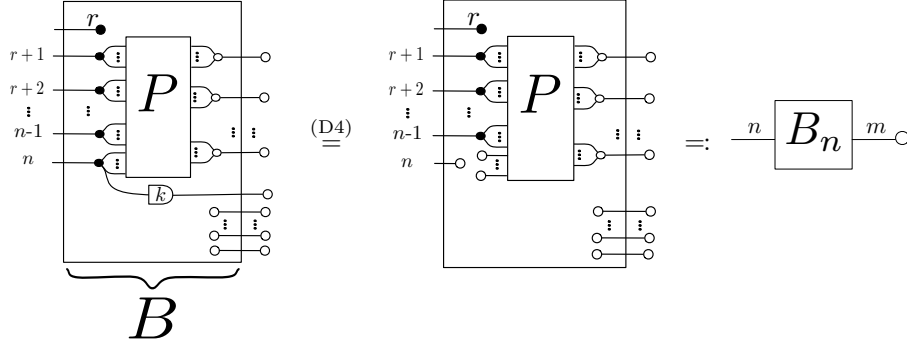


As yet another consequence of the definition of HNF, we know that, for each i with $m \geq i > f(n)$, row i only have 0 entries, allowing us to represent all the rows below $f(n)$ in the circuit above as ports on the right boundary not connected to any port on the left. Once we plug counits on the right of the circuit representing B , we trigger the “chain reaction” described in the proof of Proposition 2, which we now reproduce in circuit terms. By backward induction on i with $n \geq i > r$, we construct circuits B_n, \dots, B_{r+1} such that:

$$\text{---}_n \boxed{B} \text{---}_m \circ = \text{---}_n \boxed{B_n} \text{---}_m \circ = \dots = \text{---}_n \boxed{B_{r+1}} \text{---}_m \circ = \frac{r}{n-r} \bullet \circ$$

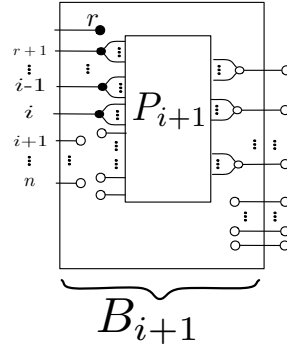
Clearly, this suffices to prove the main statement. For the base case, suppose $i = n$. The following derivation in \mathbb{HH}_R^w shows how we can “disconnect” the n -th port on the left from any port on the

right. We can then define B_n in terms of the resulting circuit.

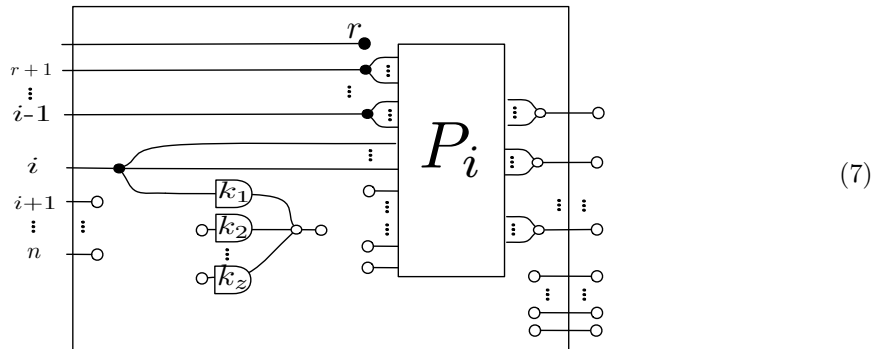


We assign the name P_n to the circuit P depicted above and proceed with the inductive step of i with $n > i > r$.

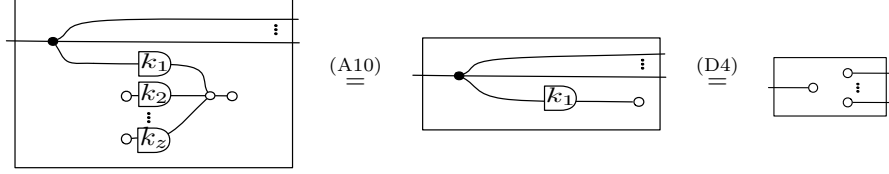
The inductive hypothesis gives us a circuit B_{i+1} as on the right. The i -th port on the left boundary corresponds to column i in B and thus it is assigned a row $f(i)$. This corresponds to the $f(i)$ -th port on the right boundary of the circuit representing B_{i+1} . By Lemma 1, such a port has no connections with ports $1, \dots, i-1$ on the left boundary. Moreover, by inductive hypothesis it also has no connections with ports $i+1, \dots, n$ on the left boundary. Therefore port $f(i)$ on the right connects only to port i on the left. These connections are part of the circuit P_{i+1} — which by inductive hypothesis only contains \square and \boxed{k} as basic components.



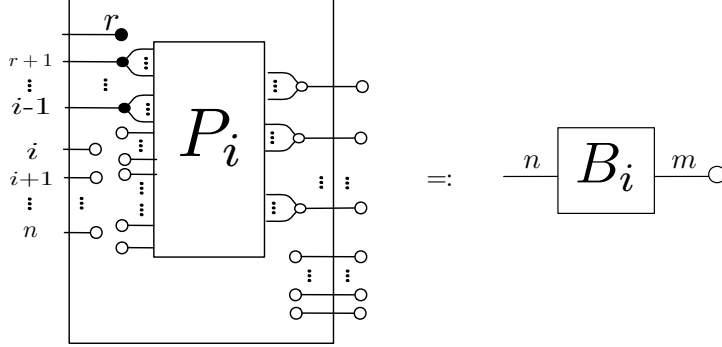
It should then be clear that we can “move port $f(i)$ towards the left side of the circuit”, isolating its connections from the others in P_{i+1} , while preserving equality in \mathbb{H}_R^w . The resulting circuit is the depicted below, where P_i results from the rearrangement of P_{i+1} in order to allow the move of port $f(i)$ towards the left side of the circuit.



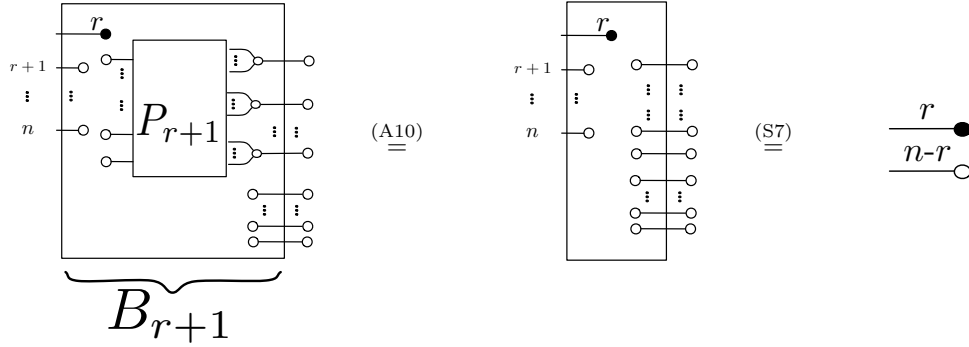
We are now allowed to consider in isolation the sub-circuit depicting the connection the connection of port i on the left with (former) port $f(i)$. It reduces as follows in \mathbb{HH}_R^w :



Thus (7) is equal to the circuit on the left below, which we use to define B_i .



Finally, at step $r+1$, our inductive construction produces a circuit as on the left below, for which we have the following derivation in \mathbb{HH}_R^w .



For the first equality, observe that by inductive hypothesis P_{r+1} is only made of basic components of the kind $\boxed{\text{gate}}$ and \boxed{k} : the white units plugged on the left boundary of P_{r+1} cancel $\boxed{\text{gate}}$ by naturality of symmetries in the symmetric monoidal category \mathbb{HH}_R^w and cancel \boxed{k} by (A10). The second equality holds by repeated application of (S7). \square

We now use Lemma 8 to formulate the soundness of kernel computation for an arbitrary R-matrix of $\text{Mat } R$.

Proposition 8. *Let $A \in \text{Mat } R[n, m]$ be a R-matrix and $\text{Ker}(A) \in \text{Mat } R[r, n]$ its kernel arrow, obtainable via the following pullback:*

$$\begin{array}{ccccc}
 & & r & & \\
 & \swarrow & \searrow & & \\
 \text{Ker}(A) & & 0_{r,0} & & \\
 & \searrow & \swarrow & & \\
 n & & 0 & & \\
 & \searrow & \swarrow & & \\
 A & & z & & 0_{0,z}
 \end{array} \tag{8}$$

Then the following equation is valid in \mathbb{H}_R^w :

$$\begin{array}{c} n \\ \text{---} \end{array} \boxed{A} \begin{array}{c} m \\ \text{---} \end{array} \circ = \begin{array}{c} n \\ \text{---} \end{array} \boxed{\text{Ker}(A)^*} \begin{array}{c} r \\ \text{---} \end{array} \bullet$$

Proof. Let $B = AU$ be the HNF of A for some invertible matrix $U: n \rightarrow n$. By the isomorphism $\text{Mat } \mathbb{R} \cong \mathbb{H}\mathbb{A}_{\mathbb{R}}$, we know that the following equation holds in $\mathbb{H}\mathbb{A}_{\mathbb{R}}$, and thus also in $\mathbb{H}\mathbb{H}_{\mathbb{R}}^w$:

$$\text{---}^n \boxed{A} \text{---}^m \circ = \text{---}^n \boxed{U^{-1}} \text{---}^n \boxed{B} \text{---}^m \circ$$

By definition the columns of matrix $\text{Ker}(A): r \rightarrow n$ yield a basis for the kernel of A . By Proposition 2, the same is true for the matrix $U_{\upharpoonright r}: r \rightarrow n$. Thus $U_{\upharpoonright r}: r \rightarrow n$ together with $0_{r,0}: r \rightarrow 0$ also serves as a pullback span in (8) and since $\mathcal{S}_{\mathbb{H}\mathbb{A}_R}(0_{r,0}) = \frac{r}{\bullet}$ we know by Lemma 7 that

$$\begin{array}{c} n \\ \text{---} \end{array} \boxed{\text{Ker}(A)^*} \begin{array}{c} r \\ \text{---} \end{array} \bullet = \begin{array}{c} n \\ \text{---} \end{array} \boxed{U \upharpoonright_r^*} \begin{array}{c} r \\ \text{---} \end{array} \bullet$$

Therefore, in order to prove our statement it suffices to show the following derivation in \mathbb{IH}_R^w .

$$\begin{array}{c}
\text{Lemma 5} \\
\begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2}
\end{array} \\
\text{Lemma 8} \\
\text{Diagram 3} \\
\text{Prop. 6} \\
\text{Diagram 4} \\
\text{Lemma 6} \\
\text{Diagram 5} \\
\text{Prop. 6} \\
\text{Diagram 6}
\end{array}$$



6.3 Proof of Proposition 7

We now have all the ingredients to provide a proof of our completeness statement, from which the characterization result of Theorem 1 follows.

Proof (Proof of Proposition 7).

Let A, B, C, D be as in the statement of Proposition 7. By the way in which pullbacks are computed in $\mathbf{Mat} \mathbf{R}$ (cf. Section 2.4), we know that $(\frac{C}{D}) = \text{Ker}(A| - B)$. Thus we have that:

$$\begin{array}{c}
\begin{array}{c} n \\ \hline \boxed{A} \end{array} \begin{array}{c} m \\ \hline \boxed{-B} \end{array} \begin{array}{c} z \\ \circ \end{array} \quad \text{Def. } \underline{\underline{\mathcal{S}_{\mathbb{H}\mathbb{A}_R}}}(A|-B) \quad \begin{array}{c} n+m \\ \hline \boxed{(A|-B)} \end{array} \begin{array}{c} z \\ \circ \end{array} \\
\\
\text{Prop. 8} \quad \underline{\underline{=}} \quad \begin{array}{c} n+m \\ \hline \boxed{\text{Ker}(A|-B)^*} \end{array} \begin{array}{c} r \\ \bullet \end{array} \\
\\
\text{Lemma 7} \quad \underline{\underline{=}} \quad \begin{array}{c} n+m \\ \hline \boxed{\left(\frac{C}{D}\right)^*} \end{array} \begin{array}{c} r \\ \bullet \end{array} \quad (9) \\
\\
\text{Def. } \underline{\underline{\mathcal{S}_{\mathbb{H}\mathbb{A}_R}}}\left(\frac{C}{D}\right) \quad \underline{\underline{=}} \quad \left(\begin{array}{c} n \\ \hline \boxed{C} \end{array} \begin{array}{c} m \\ \hline \boxed{D} \end{array} \begin{array}{c} r \\ \bullet \end{array} \right)^* \begin{array}{c} r \\ \bullet \end{array} \\
\\
\text{Prop. 6} \quad \underline{\underline{=}} \quad \begin{array}{c} n \\ \hline \boxed{C^*} \end{array} \begin{array}{c} m \\ \hline \boxed{D^*} \end{array} \begin{array}{c} r \\ \bullet \end{array}
\end{array}$$

The proof is concluded by the following derivation, yielding the desired equation in \mathbb{H}_R^w .

$$\begin{array}{c}
\begin{array}{c} n \\ \hline \boxed{A} \end{array} \begin{array}{c} z \\ \hline \boxed{B^*} \end{array} \begin{array}{c} m \\ \hline \end{array} \quad \text{Def. } \underline{\underline{(\cdot)^*}} \quad \begin{array}{c} n \\ \hline \boxed{A} \end{array} \begin{array}{c} m \\ \hline \boxed{B} \end{array} \begin{array}{c} z \\ \bullet \end{array} \begin{array}{c} m \\ \hline \end{array} \\
\\
\underline{\underline{(D2)}} \quad \begin{array}{c} n \\ \hline \boxed{A} \end{array} \begin{array}{c} m \\ \hline \boxed{B} \end{array} \begin{array}{c} z \\ \bullet \end{array} \begin{array}{c} m \\ \hline \end{array} \\
\\
\underline{\underline{(A9),(A2)}} \quad \begin{array}{c} n \\ \hline \boxed{A} \end{array} \begin{array}{c} m \\ \hline \boxed{-B} \end{array} \begin{array}{c} z \\ \circ \end{array} \begin{array}{c} m \\ \hline \end{array} \\
\\
\underline{\underline{(9)}} \quad \begin{array}{c} n \\ \hline \boxed{C^*} \end{array} \begin{array}{c} m \\ \hline \boxed{D^*} \end{array} \begin{array}{c} r \\ \bullet \end{array} \begin{array}{c} m \\ \hline \end{array} \\
\\
\text{Def. } \underline{\underline{(\cdot)^*}} \quad \begin{array}{c} n \\ \hline \boxed{C^*} \end{array} \begin{array}{c} r \\ \hline \boxed{D} \end{array} \begin{array}{c} m \\ \hline \end{array}
\end{array}$$

We detail the various derivation step. First, we can “bend” our circuit by definition of the compact-closed structure $(\cdot)^*$. Then we iteratively apply equation (D2) to turn the rightmost part of the compact-closed structure from black into white. This produces z copies of the antipode $\boxed{\blacksquare}$.

The third equality is given by iteratively applying axiom (A9) to push the antipodes in front of each scalar in circuit B , and then multiply all those scalars by the antipode value -1 using axiom (A2). As a result, we obtain the (circuit representing) the matrix $-B$. Then we can easily conclude using derivation (9). \square

7 The Cube: Rear Faces

In this section we employ the completeness result of Theorem 1 to shape the rear faces of the cube (\square) :

$$\begin{array}{ccccc}
 \mathbb{H}_R^b & \xleftarrow{[\tau_1, \tau_2]} & \mathbb{H}\mathbb{A}_R + \mathbb{H}\mathbb{A}_R^{op} & \xrightarrow{[\sigma_1, \sigma_2]} & \mathbb{H}\mathbb{H}_R^w \\
 \downarrow \mathcal{S}_{\mathbb{H}_R^b} & & \downarrow \mathcal{S}_{\mathbb{H}\mathbb{A}_R} + \mathcal{S}_{\mathbb{H}\mathbb{A}_R^{op}} & & \downarrow \mathcal{S}_{\mathbb{H}\mathbb{H}_R^w} \\
 \text{Cospan}(\text{Mat } R) & \xleftarrow{[\kappa_1, \kappa_2]} & \text{Mat } R + \text{Mat } R^{op} & \xrightarrow{[\iota_1, \iota_2]} & \text{Span}(\text{Mat } R)
 \end{array} \quad (\text{Rear})$$

In the diagram above, we define⁴

$$\kappa_1(A: n \rightarrow m) = (n \xleftarrow{id} n \xrightarrow{A} m), \quad \kappa_2(A: n \rightarrow m) = (n \xleftarrow{A} m \xrightarrow{id} m),$$

$$\iota_1(A: n \rightarrow m) = (n \xrightarrow{A} m \xleftarrow{id} m) \text{ and } \iota_2(A: n \rightarrow m) = (n \xrightarrow{id} n \xleftarrow{A} m).$$

The PROP morphisms $\sigma_1: \mathbb{H}\mathbb{A}_R \rightarrow \mathbb{H}\mathbb{H}_R^w$ and $\sigma_2: \mathbb{H}\mathbb{A}_R^{op} \rightarrow \mathbb{H}\mathbb{H}_R^w$ interpret a circuit of $\mathbb{H}\mathbb{A}_R$ ($\mathbb{H}\mathbb{A}_R^{op}$) as one of $\mathbb{H}\mathbb{H}_R^w$. The definitions of τ_1 , τ_2 and $\mathbb{H}\mathbb{H}_R^b$ will be provided in Section 7.2.

7.1 Right Rear Face: $\mathbb{H}\mathbb{H}_R^w$ and $\text{Span}(\text{Mat } R)$

We give an explicit description of the isomorphism stated in Theorem 1, in the form of a semantics $\mathcal{S}_{\mathbb{H}\mathbb{H}_R^w}: \mathbb{H}\mathbb{H}_R^w \rightarrow \text{Span}(\text{Mat } Z)$. It will be presented by induction on a circuit $c \in \mathbb{H}\mathbb{H}_R^w$, where $c \in \Sigma_{\mathbb{H}\mathbb{A}_R}$ means that c is a basic operation in the signature generating $\mathbb{H}\mathbb{A}_R$, and similarly for $c \in \Sigma_{\mathbb{H}\mathbb{A}_R^{op}}$.

$$c \mapsto \begin{cases} \kappa_1(\mathcal{S}_{\mathbb{A}\mathbb{B}}(c')) & \text{if } c = \sigma_1(c') \text{ and } c' \in \Sigma_{\mathbb{H}\mathbb{A}_R} \\ \kappa_2(\mathcal{S}_{\mathbb{A}\mathbb{B}}^{op}(c')) & \text{if } c = \sigma_2(c') \text{ and } c' \in \Sigma_{\mathbb{H}\mathbb{A}_R^{op}} \\ \mathcal{S}_{\mathbb{H}\mathbb{H}_R^w}(c_1); \mathcal{S}_{\mathbb{H}\mathbb{H}_R^w}(c_2) & \text{if } c = c_1; c_2 \\ \mathcal{S}_{\mathbb{H}\mathbb{H}_R^w}(c_1) \oplus \mathcal{S}_{\mathbb{H}\mathbb{H}_R^w}(c_2) & \text{if } c = c_1 \oplus c_2 \end{cases}$$

The semantics is well-defined as all the equations of $\mathbb{H}\mathbb{H}_R^w$ are sound w.r.t. $\mathcal{S}_{\mathbb{H}\mathbb{H}_R^w}$. Our task is now to show the following result.

Proposition 9. $\mathcal{S}_{\mathbb{H}\mathbb{H}_R^w}$ is an isomorphism of PROPs.

For this purpose, it suffices to show that $\mathcal{S}_{\mathbb{H}\mathbb{H}_R^w}$ is full and faithful. Fullness does not pose any real challenge. On the other hand, for faithfulness we need the result of Theorem 1; in particular, the following factorisation property coming from the fact that $\mathbb{H}\mathbb{H}_R^w \cong \text{Span}(\mathbb{H}\mathbb{A}_R)$:

Corollary 2. Let $c \in \mathbb{H}\mathbb{H}_R^w[n, m]$ be a circuit. Then $c = \sigma_2(c_1); \sigma_1(c_2)$ with $c_1 \in \mathbb{H}\mathbb{A}_R^{op}[n, z]$ and $c_2 \in \mathbb{H}\mathbb{A}_R[z, m]$ for some natural number z .

Proof (Proof of Proposition 9). For fullness, let $n \xleftarrow{A} z \xrightarrow{B} m$ be an arrow in $\text{Span}(\text{Mat } R)$. By fullness of $\mathcal{S}_{\mathbb{H}\mathbb{A}_R}$ there are circuits $c_1 \in \mathbb{H}\mathbb{A}_R[z, n]$ and $c_2 \in \mathbb{H}\mathbb{A}_R[z, m]$ such that $\mathcal{S}_{\mathbb{H}\mathbb{A}_R}(c_1) = A$ and $\mathcal{S}_{\mathbb{H}\mathbb{A}_R}(c_2) = B$. The following derivation shows that $n \xleftarrow{A} z \xrightarrow{B} m$ is targeted by $\sigma_2(c_1^*); \sigma_1(c_2) \in \mathbb{H}\mathbb{H}_R^w[n, m]$.

$$\begin{aligned}
 \mathcal{S}_{\mathbb{H}\mathbb{H}_R^w}(\sigma_2(c_1^*); \sigma_1(c_2)) &= \mathcal{S}_{\mathbb{H}\mathbb{H}_R^w}(\sigma_2(c_1^*); \mathcal{S}_{\mathbb{H}\mathbb{H}_R^w}(\sigma_1(c_2))) \\
 &= \kappa_2(\mathcal{S}_{\mathbb{H}\mathbb{A}_R}^{op}(c_1^*)); \kappa_2(\mathcal{S}_{\mathbb{H}\mathbb{A}_R}(c_2)) \\
 &= \kappa_2(A: n \rightarrow z); \kappa_2(B: z \rightarrow m) \\
 &= (n \xleftarrow{A} z \xrightarrow{id} z); (z \xleftarrow{id} z \xrightarrow{B} m) \\
 &= n \xleftarrow{A} z \xrightarrow{B} m.
 \end{aligned}$$

⁴ Observe that in the source of $\kappa_2: \text{Mat } R^{op} \rightarrow \text{Cospan}(\text{Mat } R)$ and $\kappa_2: \text{Mat } R^{op} \rightarrow \text{Cospan}(\text{Mat } R)$ the matrix A is seen as an element of $\text{Mat } R^{op}[n, m]$, and in their target as an element of $\text{Mat } Z[m, n]$.

It remains to show faithfulness. For this purpose, let $c \in \mathbb{I}\mathbb{H}_R^w[n, m]$ and $c' \in \mathbb{I}\mathbb{H}_R^w[n, m]$ be circuits and suppose that $\mathcal{S}_{\mathbb{I}\mathbb{H}_R^w}(c) = \mathcal{S}_{\mathbb{I}\mathbb{H}_R^w}(c')$. By Corollary 2 it follows that

$$\begin{aligned} \mathcal{S}_{\mathbb{I}\mathbb{H}_R^w}(c) &= n \xleftarrow{\mathcal{S}_{\mathbb{H}\mathbb{A}_R}(c_1^*)} z \xrightarrow{\mathcal{S}_{\mathbb{H}\mathbb{A}_R}(c_2)} m \\ \mathcal{S}_{\mathbb{I}\mathbb{H}_R^w}(c') &= n \xleftarrow{\mathcal{S}_{\mathbb{H}\mathbb{A}_R}(c_1'^*)} z \xrightarrow{\mathcal{S}_{\mathbb{H}\mathbb{A}_R}(c_2')} m \end{aligned}$$

for circuits c_1, c_1' of $\mathbb{H}\mathbb{A}_R^{op}$ and c_2, c_2' of $\mathbb{H}\mathbb{A}_R$ such that $c = \sigma_2(c_1); \sigma_1(c_2)$ and $c' = \sigma_2(c_1'); \sigma_1(c_2')$. Since $\mathcal{S}_{\mathbb{I}\mathbb{H}_R^w}(c) = \mathcal{S}_{\mathbb{I}\mathbb{H}_R^w}(c')$ are the same arrow of $\mathbf{Span}(\mathbf{Mat}\, R)$, that means they are isomorphic spans: thus there is an invertible matrix $U \in \mathbf{Mat}\, R[z, z]$ making the following diagram commute.

$$\begin{array}{ccc} & z & \\ \mathcal{S}_{\mathbb{H}\mathbb{A}_R}(c_1^*) \swarrow & \uparrow U & \searrow \mathcal{S}_{\mathbb{H}\mathbb{A}_R}(c_2) \\ n & z & m \\ \mathcal{S}_{\mathbb{H}\mathbb{A}_R}(c_1'^*) \swarrow & & \searrow \mathcal{S}_{\mathbb{H}\mathbb{A}_R}(c_2') \end{array}$$

Then by Lemma 7 we have that c and c' are equal as circuits of $\mathbb{I}\mathbb{H}_R^w$. \square

Finally, we state that, by definition, $\mathcal{S}_{\mathbb{I}\mathbb{H}_R^w}$ makes the following diagram commute, yielding the right part of (Rear).

$$\begin{array}{ccc} \mathbb{H}\mathbb{A}_R + \mathbb{H}\mathbb{A}_R^{op} & \xrightarrow{[\sigma_1, \sigma_2]} & \mathbb{I}\mathbb{H}_R^w \\ \downarrow \mathcal{S}_{\mathbb{H}\mathbb{A}_R} + \mathcal{S}_{\mathbb{H}\mathbb{A}_R}^{op} & & \downarrow \mathcal{S}_{\mathbb{I}\mathbb{H}_R^w} \\ \mathbf{Mat}\, R + \mathbf{Mat}\, R^{op} & \xrightarrow{[\kappa_1, \kappa_2]} & \mathbf{Span}(\mathbf{Mat}\, R) \end{array} \quad (\text{Rear Right})$$

7.2 Left Rear Face: $\mathbb{I}\mathbb{H}_R^b$ and $\mathbf{Cospan}(\mathbf{Mat}\, R)$

The operation of taking the transpose of a matrix yields a PROP isomorphism $(\cdot)^T: \mathbf{Mat}\, R \cong \mathbf{Mat}\, R^{op}$, defined by mapping $A: n \rightarrow m$ into $A^T: m \rightarrow n$, which is an arrow of type $n \rightarrow m$ in $\mathbf{Mat}\, R^{op}$. This also induces a PROP morphism $\mathcal{T}: \mathbf{Span}(\mathbf{Mat}\, R) \rightarrow \mathbf{Cospan}(\mathbf{Mat}\, R)$ mapping $n \xleftarrow{A} z \xrightarrow{B} m$ into $n \xrightarrow{A^T} z \xleftarrow{B^T} m$. To see that this assignment is functorial, recall from Section 2.4 that pushouts in $\mathbf{Mat}\, R$ — giving composition in $\mathbf{Cospan}(\mathbf{Mat}\, Z)$ — are calculated by transposing pullbacks of transposed matrices. In fact, because $(\cdot)^T$ is an isomorphism, also \mathcal{T} is an isomorphism.

In this section we want to provide a circuit characterization of $\mathbf{Cospan}(\mathbf{Mat}\, R)$. Since we already have such a result for $\mathbf{Span}(\mathbf{Mat}\, R)$, then our strategy will be to understand the transpose operation $(\cdot)^T$ in terms of circuits, as this will give “for free” also the syntactic PROP of $\mathbf{Cospan}(\mathbf{Mat}\, R)$. For such purpose, we first define the axiomatization that we claim to present $\mathbf{Cospan}(\mathbf{Mat}\, R)$.

Definition 8. The PROP $\mathbb{I}\mathbb{H}_R^b$ is given by quotienting $\mathbb{H}\mathbb{A}_R + \mathbb{H}\mathbb{A}_R^{op}$ out of the following equations, where k, k_1, k_2 range over R and $m = p_1 \cdot k_1 = p_2 \cdot k_2$ is the least common multiple of k_1 and k_2 .

$$\boxed{k_1 \mid k_2} = \boxed{p_1 \mid p_2} \quad (\text{T1})$$

$$\begin{array}{ccc} \boxed{\text{S-circuit}} = \boxed{\text{Circuit with two dots}} = \boxed{\text{Circuit with two dots}} & \quad & \boxed{\text{S-circuit}} = \boxed{\text{Circuit with two dots}} = \boxed{\text{Circuit with two dots}} \\ \boxed{\text{Circuit with two dots}} = \boxed{\text{Circuit with two dots}} & \quad & \boxed{\text{Circuit with two dots}} = \boxed{\text{Circuit with two dots}} \end{array}$$

$$\boxed{k \bullet} = \boxed{\bullet} \quad (\text{T2})$$

$$\boxed{\bullet \bullet} = id_0 \quad (\text{T3})$$

$$\boxed{k \bullet} = \boxed{\bullet \begin{matrix} k \\ k \end{matrix}} \quad \boxed{\bullet \bullet} = \boxed{\bullet \begin{matrix} k \\ k \end{matrix}} \bullet$$

The axioms of \mathbb{IH}_R^b are the *photographic negative* of the ones of \mathbb{IH}_R^w , that is, they are the same modulo swapping the black and white colors (and the orientation of scalar circuits). More formally, we inductively define a PROP morphism $\mathcal{N}: \mathbb{IH}_R^b \rightarrow \mathbb{IH}_R^w$ by the following mapping.

$$\begin{array}{cccc} \boxed{\bullet} \mapsto \boxed{\circ} & \boxed{\bullet} \mapsto \boxed{\circ} & \boxed{\circ} \mapsto \boxed{\bullet} & \boxed{\circ} \mapsto \boxed{\bullet} \\ \boxed{\bullet \bullet} \mapsto \boxed{\bullet \bullet} & \boxed{\bullet \bullet} \mapsto \boxed{\bullet \bullet} & \boxed{\bullet \bullet} \mapsto \boxed{\bullet \bullet} & \boxed{\bullet \bullet} \mapsto \boxed{\bullet \bullet} \\ \boxed{k} \mapsto \boxed{k} & \boxed{k} \mapsto \boxed{k} & c; c' \mapsto \mathcal{N}(c); \mathcal{N}(c') & c \oplus c' \mapsto \mathcal{N}(c) \oplus \mathcal{N}(c') \end{array}$$

The following lemma verifies that \mathcal{N} is well-defined.

Lemma 9. *For all circuits c, c' of \mathbb{IH}_R^b , $c = c'$ in \mathbb{IH}_R^b if and only if $\mathcal{N}(c) = \mathcal{N}(c')$ in \mathbb{IH}_R^w .*

Proof. By construction, the equations presenting \mathbb{IH}_R^w are the image under \mathcal{N} of the equations presenting \mathbb{IH}_R^b . Thus the statement is also true for all the derived laws of the two theories. \square

Lemma 10. *\mathcal{N} is an isomorphism of PROPs.*

Proof. Fullness of \mathcal{N} is easily verified by induction on $c \in \mathbb{IH}_R^w$ and faithfulness follows by the “only if” direction of Lemma 9. Since \mathbb{IH}_R^w and \mathbb{IH}_R^b have the same objects, this suffices to verify the statement. \square

We can now define the desired isomorphism between \mathbb{IH}_R^b and $\text{Cospan}(\text{Mat } \mathbb{Z})$ as the composite

$$\mathbb{IH}_R^b \xrightarrow{\mathcal{N}} \mathbb{IH}_R^w \xrightarrow{\mathcal{S}_{\mathbb{IH}_R^w}} \text{Span}(\text{Mat } \mathbb{Z}) \xrightarrow{\mathcal{T}} \text{Cospan}(\text{Mat } \mathbb{Z}) .$$

In fact, we aim at presenting such correspondence in a more direct way. For this purpose, let $\tau_1: \mathbb{HA}_R \rightarrow \mathbb{IH}_R^b$ and $\tau_2: \mathbb{HA}_R^{op} \rightarrow \mathbb{IH}_R^b$ be given by interpreting a circuit of \mathbb{HA}_R (\mathbb{HA}_R^{op}) as one of \mathbb{IH}_R^b . Similarly to what we did for \mathbb{IH}_R^w and $\text{Span}(\text{Mat } \mathbb{Z})$, we define a semantics $\mathcal{S}_{\mathbb{IH}_R^b}: \mathbb{IH}_R^b \rightarrow \text{Cospan}(\text{Mat } \mathbb{R})$ by induction on $c \in \mathbb{IH}_R^b$ as follows,

$$c \mapsto \begin{cases} \iota_1(\mathcal{S}_{\mathbb{AB}}(c')) & \text{if } c = \tau_1(c') \text{ and } c' \in \Sigma_{\mathbb{HA}_R} \\ \iota_2(\mathcal{S}_{\mathbb{AB}}^{op}(c')) & \text{if } c = \tau_2(c') \text{ and } c' \in \Sigma_{\mathbb{HA}_R^{op}} \\ \mathcal{S}_{\mathbb{IH}_R^b}(c_1); \mathcal{S}_{\mathbb{IH}_R^b}(c_2) & \text{if } c = c_1; c_2 \\ \mathcal{S}_{\mathbb{IH}_R^b}(c_1) \oplus \mathcal{S}_{\mathbb{IH}_R^b}(c_2) & \text{if } c = c_1 \oplus c_2 \end{cases}$$

The semantics is well-defined as all the equations of \mathbb{IH}_R^b are sound w.r.t. $\mathcal{S}_{\mathbb{IH}_R^b}$.

Proposition 10. *$\mathcal{S}_{\mathbb{IH}_R^b}$ is an isomorphism of PROPs.*

Proof. It suffices to show that $\mathcal{S}_{\mathbb{IH}_R^b} = \mathcal{N}; \mathcal{S}_{\mathbb{IH}_R^w}; \mathcal{T}$. This can be easily verified by induction on

$c \in \mathbb{IH}_R^b$. For instance, $\mathcal{S}_{\mathbb{IH}_R^b}$ maps $\boxed{\bullet}: 2 \rightarrow 1$ into $2 \xrightarrow{id} 2 \xleftarrow{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} 1$. Instead $\mathcal{N}; \mathcal{S}_{\mathbb{IH}_R^w}; \mathcal{T}$ maps $\boxed{\bullet}$ first to $\boxed{\circ}$, then to $2 \xleftarrow{id} 2 \xrightarrow{\begin{pmatrix} 1 & 1 \end{pmatrix}} 1$ and finally to $2 \xrightarrow{id} 2 \xleftarrow{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} 1$. \square

Also, by definition, $\mathcal{S}_{\mathbb{IH}_R^b}$ makes the following diagram commute, yielding the left part of (Rear).

$$\begin{array}{ccc} \mathbb{IH}_R^b & \xleftarrow{[\tau_1, \tau_2]} & \mathbb{HA}_R + \mathbb{HA}_R^{op} \\ \mathcal{S}_{\mathbb{IH}_R^b} \downarrow & & \downarrow \mathcal{S}_{\mathbb{HA}_R} + \mathcal{S}_{\mathbb{HA}_R}^{op} \\ \text{Span}(\text{Mat } \mathbb{R}) & \xleftarrow{[\iota_1, \iota_2]} & \text{Mat } \mathbb{R} + \text{Mat } \mathbb{R}^{op} \end{array} \quad (\text{Rear Left})$$

8 The Cube: Top Face

The theory \mathbb{IH}_R is obtained as the sum of theories \mathbb{IH}_R^w and \mathbb{IH}_R^b . As we want to identify the basic operations of $\mathbb{HA}_R + \mathbb{HA}_R^{op}$ on which both \mathbb{IH}_R^w and \mathbb{IH}_R^b are based, we formally define it as the following pushout.

$$\begin{array}{ccccc}
& & \mathbb{H}\mathbb{A}_R + \mathbb{H}\mathbb{A}_R^{op} & & \\
& \swarrow [\sigma_1, \sigma_2] & \downarrow [\varphi_1, \varphi_2] & \searrow [\tau_1, \tau_2] & \\
\mathbb{H}\mathbb{H}_R^w & & & & \mathbb{H}\mathbb{H}_R^b \quad (\text{Top}) \\
& \searrow \theta & & \swarrow \Lambda & \\
& & \mathbb{H}\mathbb{H}_R & &
\end{array}$$

The PROP morphism $[\varphi_1, \varphi_2]: \mathbb{H}\mathbb{A}_R + \mathbb{H}\mathbb{A}_R^{op} \rightarrow \mathbb{I}\mathbb{H}_R$ is defined by commutativity of the diagram. The PROP morphism Θ quotients $\mathbb{I}\mathbb{H}_R^b$ by the equations of $\mathbb{I}\mathbb{H}_R^b$ and Λ quotients $\mathbb{I}\mathbb{H}_R^b$ by the ones of $\mathbb{I}\mathbb{H}_R^w$. We can give a presentation of the resulting theory $\mathbb{I}\mathbb{H}_R$ as follows.

Definition 9. The PROP $\mathbb{I}\mathbb{H}_{\mathbb{R}}$ is given by quotienting $\mathbb{H}\mathbb{A}_{\mathbb{R}} + \mathbb{H}\mathbb{A}_{\mathbb{R}}^{op}$ out of the following equations, where k, k_1, k_2 range over \mathbb{R} and $m = p_1 \cdot k_1 = p_2 \cdot k_2$ is the least common multiple of k_1 and k_2 .

Figure 1 displays 12 equations representing the reduction of various string diagrams to simpler forms. The diagrams use boxes, circles, and lines to represent mathematical operations. Some boxes are labeled with k_1 , k_2 , p_1 , p_2 , or k . Some circles contain black dots. The equations are arranged in two columns and six rows.

- Equation 1 (Top Left): A box containing two circles labeled k_1 and k_2 in series is equal to a box containing two circles labeled p_1 and p_2 in series.
- Equation 2 (Top Right): A box containing two circles labeled k_1 and k_2 in parallel is equal to a box containing two circles labeled p_1 and p_2 in parallel.
- Equation 3 (Second Row Left): A box containing two circles in series, with the first circle having an incoming line from the left and the second circle having an outgoing line to the right, is equal to a box containing two circles in series, with the first circle having an incoming line from the left and the second circle having an outgoing line to the right.
- Equation 4 (Second Row Right): A box containing two circles in series, with the first circle having an incoming line from the left and the second circle having an outgoing line to the right, is equal to a box containing two circles in series, with the first circle having an incoming line from the left and the second circle having an outgoing line to the right.
- Equation 5 (Third Row Left): A box containing a circle with an incoming line from the left and an outgoing line to the right is equal to a box containing a circle with an incoming line from the left and an outgoing line to the right.
- Equation 6 (Third Row Right): A box containing a circle with an incoming line from the left and an outgoing line to the right is equal to a box containing a circle with an incoming line from the left and an outgoing line to the right.
- Equation 7 (Fourth Row Left): A box containing a circle with an incoming line from the left and an outgoing line to the right is equal to a box containing a circle with an incoming line from the left and an outgoing line to the right.
- Equation 8 (Fourth Row Right): A box containing a circle with an incoming line from the left and an outgoing line to the right is equal to a box containing a circle with an incoming line from the left and an outgoing line to the right.
- Equation 9 (Fifth Row Left): A box containing a circle with an incoming line from the left and an outgoing line to the right is equal to a box containing a circle with an incoming line from the left and an outgoing line to the right.
- Equation 10 (Fifth Row Right): A box containing a circle with an incoming line from the left and an outgoing line to the right is equal to a box containing a circle with an incoming line from the left and an outgoing line to the right.
- Equation 11 (Bottom Left): A box containing a circle with an incoming line from the left and an outgoing line to the right is equal to id_0 .
- Equation 12 (Bottom Right): A box containing a circle with an incoming line from the left and an outgoing line to the right is equal to id_0 .

Observe that \mathbb{H}_R is presented by the axioms of \mathbb{H}_R^w plus (T1), (T2) and (T3) from the presentation of \mathbb{H}_R^b . Indeed, the other axioms of \mathbb{H}_R^b are derived laws (D1), (D2), (D5) and (D6) in \mathbb{H}_R^w . Since \mathbb{H}_R is both a quotient of $\mathbb{H}_R^w \cong \text{Span}(\mathbb{H}\mathbb{A}_R)$ and of $\mathbb{H}_R^b \cong \text{Cospan}(\mathbb{H}\mathbb{A}_R)$, it inherits their factorisation property.

Theorem 2 (Factorisation of \mathbb{IH}_R). *Let $c \in \mathbb{IH}_R[n, m]$ be a circuit.*

- There exist $c_1 \in \mathbb{H}_{\mathbb{R}}^{op}[n, z]$ and $c_2 \in \mathbb{H}_{\mathbb{R}}[z, m]$ such that $c = \varphi_2(c_1); \varphi_1(c_2)$, for some z .
– There exist $c_3 \in \mathbb{H}_{\mathbb{R}}[n, z']$ and $c_4 \in \mathbb{H}_{\mathbb{R}}^{op}[z', m]$ such that $c = \varphi_1(c_3); \varphi_2(c_4)$, for some z' .

Proof. The first characterisation follows by Corollary 2. Since $\mathbb{IH}_{\mathbb{R}}^b$ has been shown to be isomorphic to $\mathbf{Cospan}(\mathbb{HA}_{\mathbb{R}})$, then a result analogous to Corollary 2 also holds for $\mathbb{IH}_{\mathbb{R}}^b$, yielding the second characterisation statement. \square

9 The Cube: Bottom Face

Let \mathbb{SV}_k be the PROP with arrows $n \rightarrow m$ vector subspaces of $k^n \times k^m$, considered as a k -vector space. Composition is relational: given $V: n \rightarrow z$, $W: z \rightarrow m$,

$$(\mathbf{x}, \mathbf{z}) \in V; W \quad \Leftrightarrow \quad \exists \mathbf{y}. (\mathbf{x}, \mathbf{y}) \in V \wedge (\mathbf{y}, \mathbf{z}) \in W$$

In this section we show that the following diagram, which is the bottom face of the cube (\mathcal{A}) , is a pushout in **PROP**.

$$\begin{array}{ccc} \text{Mat } R + \text{Mat } R^{op} & \xrightarrow{[\kappa_1, \kappa_2]} & \text{Span}(\text{Mat } R) \\ \downarrow [\iota_1, \iota_2] & & \downarrow \Phi \\ \text{Cospan}(\text{Mat } R) & \xrightarrow{\Psi} & \mathbb{SV}_k \end{array} \quad (\text{Bot})$$

For the definition of κ_1 , κ_2 , ι_1 and ι_2 see the beginning of Section 7. The definitions of Φ and Ψ follow in the proceeding sections.

9.1 Definition of $\Phi: \text{Span}(\text{Mat } R) \rightarrow \mathbb{SV}_k$

Let $\Phi(n \xleftarrow{A} z \xrightarrow{B} m)$ be defined to be the subspace

$$\{ (\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \in k^n, \mathbf{y} \in k^m, \exists \mathbf{z} \in k^z. A\mathbf{z} = \mathbf{x} \wedge B\mathbf{z} = \mathbf{y} \}.$$

Lemma 11. Φ is a PROP morphism.

Proof. We must verify that Φ preserves composition. In the diagram below let the centre square be a pullback diagram.

$$\begin{array}{ccccc} & & r & & \\ & F'_2 \swarrow & & \searrow G'_1 & \\ & z_1 & & z_2 & \\ F_1 \swarrow & & G_1 \searrow & F_2 \swarrow & G_2 \searrow \\ n & & z & & m \end{array}$$

By definition of composition in $\text{Span}(\text{Mat } R)$, $(\xleftarrow{F_1} \xrightarrow{G_1}); (\xleftarrow{F_2} \xrightarrow{G_2}) = \xleftarrow{F_1 F'_2} \xrightarrow{G_2 G'_1}$.

Now, by definition, if $(\mathbf{x}, \mathbf{z}) \in \Phi(\xleftarrow{F_1 F'_2} \xrightarrow{G_2 G'_1})$ then $\exists \mathbf{w}$ with $\mathbf{x} = F_1 F'_2 \mathbf{w}$ and $\mathbf{z} = G_2 G'_1 \mathbf{w}$. Since the square commutes, we have $(\mathbf{x}, \mathbf{z}) \in \Phi(\xleftarrow{F_1} \xrightarrow{G_1}); \Phi(\xleftarrow{F_2} \xrightarrow{G_2})$.

Conversely, if $(\mathbf{x}, \mathbf{z}) \in \Phi(\xleftarrow{F_1} \xrightarrow{G_1}); \Phi(\xleftarrow{F_2} \xrightarrow{G_2})$ then for some \mathbf{y} we must have $(\mathbf{x}, \mathbf{y}) \in \Phi(\xleftarrow{F_1} \xrightarrow{G_1})$ and $(\mathbf{y}, \mathbf{z}) \in \Phi(\xleftarrow{F_2} \xrightarrow{G_2})$. Thus there exists \mathbf{u} with $\mathbf{x} = F_1 \mathbf{u}$ and $\mathbf{y} = G_1 \mathbf{u}$ and there exists \mathbf{v} with $\mathbf{y} = F_2 \mathbf{v}$ and $\mathbf{z} = G_2 \mathbf{v}$. Since the square is also a pullback in $\text{Mat } k$, it translates to a pullback diagram in $\text{FMod } k$: it follows that $\exists \mathbf{w}$ with $F'_2 \mathbf{w} = \mathbf{u}$ and $G'_1 \mathbf{w} = \mathbf{v}$: thus $(\mathbf{x}, \mathbf{z}) \in \Phi(\xleftarrow{F_1 F'_2} \xrightarrow{G_2 G'_1})$; $(\xleftarrow{F_2} \xrightarrow{G_2})$). This completes the proof. \square

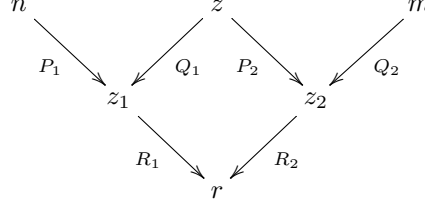
9.2 Definition of $\Psi: \text{Cospan}(\text{Mat } R) \rightarrow \mathbb{SV}_k$

Let $\Psi(n \xrightarrow{A} z \xleftarrow{B} m)$ be defined to be the subspace

$$\{ (\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \in k^n, \mathbf{y} \in k^m, A\mathbf{x} = B\mathbf{y} \}$$

Lemma 12. Ψ is a PROP morphism.

Proof. We must verify that Ψ preserves composition. Let the square in the diagram below be a pushout in $\mathbf{Mat R}$. By definition of composition in $\mathbf{Cospan}(\mathbf{Mat R})$ we have $(\xrightarrow{P_1} \xleftarrow{Q_1}); (\xrightarrow{P_2} \xleftarrow{Q_2}) = \xrightarrow{R_1 P_1} \xleftarrow{R_2 Q_2}$.



Consider $(\mathbf{x}, \mathbf{z}) \in \Psi(\xrightarrow{R_1 P_1} \xleftarrow{R_2 Q_2})$. Then $R_1 P_1 \mathbf{x} = R_2 Q_2 \mathbf{z} = \mathbf{y} \in \mathbf{k}^r$. Since the pushout diagram maps to a pushout diagram in $\mathbf{FMod k}$, we can use the conclusions of Lemma 3 to obtain $\mathbf{y} \in \mathbf{k}^z$ such that $Q_1 \mathbf{y} = P_1 \mathbf{x}$ and $P_2 \mathbf{y} = Q_2 \mathbf{z}$. In other words, we have $(\mathbf{x}, \mathbf{y}) \in \Psi(\xrightarrow{P_1} \xleftarrow{Q_1})$ and $(\mathbf{y}, \mathbf{z}) \in \Psi(\xrightarrow{P_2} \xleftarrow{Q_2})$, meaning that $(\mathbf{x}, \mathbf{z}) \in \Psi(\xrightarrow{P_1} \xleftarrow{Q_1}); \Psi(\xrightarrow{P_2} \xleftarrow{Q_2})$.

Conversely if $(\mathbf{x}, \mathbf{z}) \in \Psi(\xrightarrow{P_1} \xleftarrow{Q_1}); \Psi(\xrightarrow{P_2} \xleftarrow{Q_2})$ then $\exists \mathbf{y} \in \mathbf{k}^z$ such that $(\mathbf{x}, \mathbf{y}) \in \Psi(\xrightarrow{P_1} \xleftarrow{Q_1})$ and $(\mathbf{y}, \mathbf{z}) \in \Psi(\xrightarrow{P_2} \xleftarrow{Q_2})$. It follows that $R_1 P_1 \mathbf{x} = R_1 Q_1 \mathbf{y} = R_2 P_2 \mathbf{y} = R_2 Q_2 \mathbf{z}$ and thus $(\mathbf{x}, \mathbf{z}) \in \Psi(\xrightarrow{R_1 P_1} \xleftarrow{R_2 Q_2})$ as required. \square

9.3 Properties of (Bot)

Lemma 13. (Bot) *commutes.*

Proof. It suffices to show that it commutes on the two injections into $\mathbf{Mat R} + \mathbf{Mat R}^{op}$. This means that we have to show, for any $A: n \rightarrow m$ in $\mathbf{Mat R}$, that

$$\Phi(\xleftarrow{id} \xrightarrow{A}) = \Psi(\xrightarrow{A} \xleftarrow{id})$$

and

$$\Phi(\xrightarrow{A} \xleftarrow{id}) = \Psi(\xleftarrow{id} \xrightarrow{A}).$$

These are clearly symmetric, so it is enough to check one. But this follows directly from the definition of Φ and Ψ :

$$\Phi(\xleftarrow{id} \xrightarrow{A}) = \{ (\mathbf{x}, \mathbf{y}) \mid A\mathbf{x} = \mathbf{y} \} = \Psi(\xrightarrow{A} \xleftarrow{id})$$

\square

Lemma 14. *Given an arbitrary PROP \mathbb{X} and a commutative diagram*

$$\begin{array}{ccc} \mathbf{Mat R} + \mathbf{Mat R}^{op} & \xrightarrow{[\kappa_1, \kappa_2]} & \mathbf{Span}(\mathbf{Mat R}) \\ \downarrow [\iota_1, \iota_2] & & \downarrow \Gamma \\ \mathbf{Cospan}(\mathbf{Mat R}) & \xrightarrow{\Delta} & \mathbb{X} \end{array} \quad (\dagger)$$

consider the following diagram in $\mathbf{Mat R}$:

$$\begin{array}{ccc} & \xrightarrow{G} & \\ F \downarrow & & \downarrow Q \\ & \xrightarrow{P} & \end{array} \quad (\star)$$

- (i) if (\star) is a pushout diagram then $\Gamma(\xleftarrow{F} \xrightarrow{G}) = \Delta(\xrightarrow{P} \xleftarrow{Q})$.
- (ii) if (\star) is a pullback diagram then $\Gamma(\xleftarrow{F} \xrightarrow{G}) = \Delta(\xrightarrow{P} \xleftarrow{Q})$.

- (iii) if $\langle \xleftarrow{F_1} \xrightarrow{G_1} \rangle$ and $\langle \xleftarrow{F_2} \xrightarrow{G_2} \rangle$ have the same pushout in $\mathbf{Mat} \mathbf{R}$ then $\Gamma(\langle \xleftarrow{F_1} \xrightarrow{G_1} \rangle) = \Gamma(\langle \xleftarrow{F_2} \xrightarrow{G_2} \rangle)$.
(iv) if $\langle \xrightarrow{P_1} \xleftarrow{Q_1} \rangle$ and $\langle \xrightarrow{P_2} \xleftarrow{Q_2} \rangle$ have the same pullback in $\mathbf{Mat} \mathbf{R}$ then $\Delta(\langle \xrightarrow{P_1} \xleftarrow{Q_1} \rangle) = \Delta(\langle \xrightarrow{P_2} \xleftarrow{Q_2} \rangle)$.

Proof. (i) Suppose that $\langle \xrightarrow{P} \xleftarrow{Q} \rangle$ is the cospan obtained by pushing out $\langle \xleftarrow{F} \xrightarrow{G} \rangle$ in $\mathbf{Mat} \mathbf{R}$. Then

$$\begin{aligned} \Gamma(\langle \xrightarrow{P} \xleftarrow{Q} \rangle) &= \Gamma(\kappa_2 F; \kappa_1 G) \\ &= \Gamma(\kappa_2 F); \Gamma(\kappa_1 G) \\ &= \Delta(\iota_2 F); \Delta(\iota_1 G) \\ &= \Delta(\iota_2 F; \iota_1 G) \\ &= \Delta(\langle \xrightarrow{P} \xleftarrow{Q} \rangle) \end{aligned}$$

(ii) Suppose that $\langle \xleftarrow{F} \xrightarrow{G} \rangle$ is the span obtained by pulling back $\langle \xrightarrow{P} \xleftarrow{Q} \rangle$. Then

$$\begin{aligned} \Delta(\langle \xrightarrow{P} \xleftarrow{Q} \rangle) &= \Delta(\iota_1 P; \iota_2 Q) \\ &= \Delta \iota_1 P; \Delta \iota_2 Q \\ &= \Gamma \kappa_1 P; \Gamma \kappa_2 Q \\ &= \Gamma(\kappa_1 P; \kappa_2 Q) \\ &= \Gamma(\langle \xleftarrow{F} \xrightarrow{G} \rangle) \end{aligned}$$

(iii) Suppose that $\langle \xrightarrow{P} \xleftarrow{Q} \rangle$ is the cospan obtained by pushing out $\langle \xleftarrow{F_1} \xrightarrow{G_1} \rangle$ and $\langle \xleftarrow{F_2} \xrightarrow{G_2} \rangle$. Using (i) we get $\Gamma(\langle \xleftarrow{F_1} \xrightarrow{G_1} \rangle) = \Delta(\langle \xrightarrow{P} \xleftarrow{Q} \rangle) = \Gamma(\langle \xleftarrow{F_2} \xrightarrow{G_2} \rangle)$. The proof of (iv) is similar and uses (ii).

Lemma 15. *The following are equivalent*

- (i) $n \xrightarrow{P_1} z_1 \xleftarrow{Q_1} m$ and $n \xrightarrow{P_2} z_2 \xleftarrow{Q_2} m$ have the same pullback in $\mathbf{Mat} \mathbf{R}$.
(ii) $\Psi(\langle \xrightarrow{P_1} \xleftarrow{Q_1} \rangle) = \Psi(\langle \xrightarrow{P_2} \xleftarrow{Q_2} \rangle)$.

Proof. The conclusions of Lemmas 13 and 14 give that (i) \Rightarrow (ii). It thus suffices to show that (ii) \Rightarrow (i). Indeed, suppose that $\Psi(\langle \xrightarrow{P_1} \xleftarrow{Q_1} \rangle) = \Psi(\langle \xrightarrow{P_2} \xleftarrow{Q_2} \rangle)$. In particular on elements $\mathbf{x} \in \mathbf{R}^n$, $\mathbf{y} \in \mathbf{R}^m$ we have $P_1 \mathbf{x} = Q_1 \mathbf{y}$ iff $P_2 \mathbf{x} = Q_2 \mathbf{y}$ (*). Compute the following pullbacks in $\mathbf{Mat} \mathbf{R}$:

$$\begin{array}{ccc} r_1 & \xrightarrow{F_1} & m \\ G_1 \downarrow & & \downarrow Q_1 \\ n & \xrightarrow{P_1} & z_1 \end{array} \quad \begin{array}{ccc} r_2 & \xrightarrow{F_2} & m \\ G_2 \downarrow & & \downarrow Q_2 \\ n & \xrightarrow{P_2} & z_2 \end{array}$$

By (*) we can conclude that $P_1 G_2 = Q_1 F_2$ and $P_2 G_1 = Q_2 F_1$. This, using the universal property of pullbacks, implies that the spans $\langle \xleftarrow{G_1} \xrightarrow{F_1} \rangle$ and $\langle \xleftarrow{G_2} \xrightarrow{F_2} \rangle$ are isomorphic.

Lemma 16. *The following are equivalent*

- (i) $n \xleftarrow{F_1} z_1 \xrightarrow{G_1} m$ and $n \xleftarrow{F_2} z_2 \xrightarrow{G_2} m$ have the same pushout in $\mathbf{Mat} \mathbf{R}$
(ii) $\Phi(\langle \xleftarrow{F_1} \xrightarrow{G_1} \rangle) = \Phi(\langle \xleftarrow{F_2} \xrightarrow{G_2} \rangle)$.

Proof. The conclusions of Lemmas 13 and 14 again give us that (i) \Rightarrow (ii). It thus suffices to show that (ii) \Rightarrow (i).

Assume $\Phi(\langle \xleftarrow{F_1} \xrightarrow{G_1} \rangle) = \Phi(\langle \xleftarrow{F_2} \xrightarrow{G_2} \rangle)$. Compute the following pushouts in $\mathbf{Mat} \mathbf{R}$:

$$\begin{array}{ccc} z_1 & \xrightarrow{F_1} & n \\ G_1 \downarrow & & \downarrow Q_1 \\ m & \xrightarrow{P_1} & r_1 \end{array} \quad \begin{array}{ccc} z_2 & \xrightarrow{F_2} & n \\ G_2 \downarrow & & \downarrow Q_2 \\ m & \xrightarrow{P_2} & r_2 \end{array}$$

By the conclusion of Lemma 14, we have $\Psi(\frac{P_1}{\rightarrow}\frac{Q_1}{\leftarrow}) = \Psi(\frac{P_2}{\rightarrow}\frac{Q_2}{\leftarrow})$. Applying the conclusion of Lemma 15, $\frac{P_1}{\rightarrow}\frac{Q_1}{\leftarrow}$ and $\frac{P_2}{\rightarrow}\frac{Q_2}{\leftarrow}$ have the same pullback span. Call this span $\frac{A}{\leftarrow}\frac{B}{\rightarrow}$. Then both $\frac{P_1}{\rightarrow}\frac{Q_1}{\leftarrow}$ and $\frac{P_2}{\rightarrow}\frac{Q_2}{\leftarrow}$ are the pushout cospan of $\frac{A}{\leftarrow}\frac{B}{\rightarrow}$, thus they must be isomorphic. \square

Lemma 17. $\Phi: \text{Span}(\text{Mat } R) \rightarrow \mathbb{S}\mathbb{V}_k$ and $\Psi: \text{Cospan}(\text{Mat } R) \rightarrow \mathbb{S}\mathbb{V}_k$ are both full.

Proof. Take any subspace $S: n \rightarrow m$ in $\mathbb{S}\mathbb{V}_k$. Picking any finite basis (say, of size r) for this subspace and multiplying out fractions gives us a finite set of elements in R^{n+m} . In the obvious way, this yields

$$n \xleftarrow{S_1} r \xrightarrow{S_2} m$$

in $\text{Span}(\text{Mat } R)$ with $\Phi(\frac{S_1}{\leftarrow}\frac{S_2}{\rightarrow}) = S$. Thus Φ is full. Let $\frac{R_1}{\rightarrow}\frac{R_2}{\leftarrow}$ be the cospan obtained from pushing out $\frac{S_1}{\leftarrow}\frac{S_2}{\rightarrow}$ in $\text{Mat } R$. By the conclusion of Lemma 14, $\Psi(\frac{R_1}{\rightarrow}\frac{R_2}{\leftarrow}) = \Phi(\frac{S_1}{\leftarrow}\frac{S_2}{\rightarrow}) = S$, which shows that also Ψ is full. \square

Theorem 3. (Bot) is a pushout of PROPs.

Proof. Suppose that we have a commutative diagram of PROP morphisms as in (†). By the conclusions of Lemma 17 it suffices to show that there exists a PROP morphism $\Theta: \mathbb{S}\mathbb{V}_k \rightarrow \mathbb{X}$ with $\Theta\Phi = \Gamma$ and $\Theta\Psi = \Delta$ – uniqueness is automatic by fullness of Φ (or of Ψ).

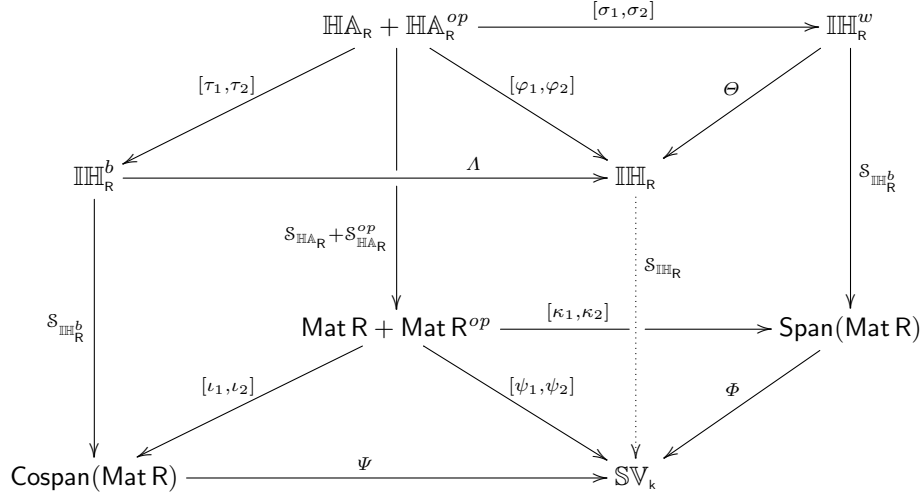
Given a subspace $S: n \rightarrow m$, by Lemma 17 there exists a span $\frac{S_1}{\leftarrow}\frac{S_2}{\rightarrow}$ with $\Phi(\frac{S_1}{\leftarrow}\frac{S_2}{\rightarrow}) = S$. We let $\Theta(S) = \Gamma(\frac{S_1}{\leftarrow}\frac{S_2}{\rightarrow})$. This is well-defined: if $\frac{S'_1}{\leftarrow}\frac{S'_2}{\rightarrow}$ is another span with $\Phi(\frac{S'_1}{\leftarrow}\frac{S'_2}{\rightarrow}) = S$ then applying the conclusions of Lemma 16 gives us that $\frac{S_1}{\leftarrow}\frac{S_2}{\rightarrow}$ and $\frac{S'_1}{\leftarrow}\frac{S'_2}{\rightarrow}$ have the same pushout in $\text{Mat } R$. Now the conclusions of Lemma 14 give us that $\Gamma(\frac{S_1}{\leftarrow}\frac{S_2}{\rightarrow}) = \Gamma(\frac{S'_1}{\leftarrow}\frac{S'_2}{\rightarrow})$. This argument also shows that, generally, $\Theta\Phi = \Gamma$. Finally, Θ preserves composition:

$$\begin{aligned} \Theta(R; S) &= \Theta(\Phi(\frac{R_1}{\rightarrow}\frac{R_2}{\leftarrow}); \Phi(\frac{S_1}{\leftarrow}\frac{S_2}{\rightarrow})) \\ &= \Theta(\Phi((\frac{R_1}{\rightarrow}\frac{R_2}{\leftarrow}); (\frac{S_1}{\leftarrow}\frac{S_2}{\rightarrow}))) \\ &= \Gamma((\frac{R_1}{\rightarrow}\frac{R_2}{\leftarrow}); (\frac{S_1}{\leftarrow}\frac{S_2}{\rightarrow})) \\ &= \Gamma(\frac{R_1}{\rightarrow}\frac{R_2}{\leftarrow}); \Gamma(\frac{S_1}{\leftarrow}\frac{S_2}{\rightarrow}) \\ &= \Theta(R); \Theta(S). \end{aligned}$$

It is also easy to show that $\Theta\Psi = \Delta$: given a cospan $\frac{F}{\rightarrow}\frac{G}{\leftarrow}$ let $\frac{P}{\leftarrow}\frac{Q}{\rightarrow}$ be its pullback span in $\text{Mat } R$. Using the conclusions of Lemma 14, $\Delta(\frac{F}{\rightarrow}\frac{G}{\leftarrow}) = \Gamma(\frac{P}{\leftarrow}\frac{Q}{\rightarrow}) = \Theta\Phi(\frac{P}{\leftarrow}\frac{Q}{\rightarrow}) = \Theta\Psi(\frac{F}{\rightarrow}\frac{G}{\leftarrow})$. \square

10 The Cube Rebuilt

We are now in position to patch together all the faces that we worked out in the previous sections to form the cube (\boxplus) .



The top face is the pushout presented in Section 8, while the bottom face is the pushout presented in Section 9. We also draw functors $[\varphi_1, \varphi_2]: \mathbb{H}\mathbb{A}_R + \mathbb{H}\mathbb{A}_R^{op} \rightarrow \mathbb{I}\mathbb{H}_R$ and $[\psi_1, \psi_2]: \mathbf{Mat}\, R + \mathbf{Mat}\, R^{op} \rightarrow \mathbb{S}\mathbb{V}_k$ defined by commutativity of the top and the bottom face diagrams. Since the rear faces commute (*cf.* Section 7) there is a unique PROP morphism $\mathcal{S}_{\mathbb{I}\mathbb{H}_R}: \mathbb{I}\mathbb{H}_R \rightarrow \mathbb{S}\mathbb{V}_k$ making the front faces commute, given by universal property of $\mathbb{I}\mathbb{H}_R$. Moreover, $\mathcal{S}_{\mathbb{I}\mathbb{H}_R}$ is an isomorphism because the other edges of the cube — $\mathcal{S}_{\mathbb{H}\mathbb{A}_R} + \mathcal{S}_{\mathbb{H}\mathbb{A}_R}^{op}$, $\mathcal{S}_{\mathbb{I}\mathbb{H}_R^w}$ and $\mathcal{S}_{\mathbb{I}\mathbb{H}_R^b}$ — have been proven to be PROP isomorphisms. Commutativity of all the faces of the cube yields also commutativity of the “section”:

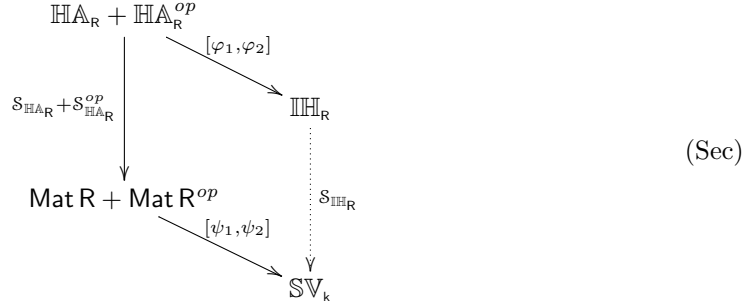


Diagram (Sec) provides us a recipe for an inductive presentation of $\mathcal{S}_{\mathbb{I}\mathbb{H}_R}$, for $c \in \mathbb{I}\mathbb{H}_R$, similarly to what we did for $\mathcal{S}_{\mathbb{I}\mathbb{H}_R^b}$ and $\mathcal{S}_{\mathbb{I}\mathbb{H}_R^w}$ in the previous sections:

$$c \mapsto \begin{cases} \psi_1(\mathcal{S}_{\mathbb{H}\mathbb{A}_R}(c')) & \text{if } c = \varphi_1(c') \text{ and } c' \in \Sigma_{\mathbb{H}\mathbb{A}_R} \\ \psi_2(\mathcal{S}_{\mathbb{H}\mathbb{A}_R}^{op}(c')) & \text{if } c = \varphi_2(c') \text{ and } c' \in \Sigma_{\mathbb{H}\mathbb{A}_R}^{op} \\ \mathcal{S}_{\mathbb{I}\mathbb{H}_R}(c_1); \mathcal{S}_{\mathbb{I}\mathbb{H}_R}(c_2) & \text{if } c = c_1; c_2 \\ \mathcal{S}_{\mathbb{I}\mathbb{H}_R}(c_1) \oplus \mathcal{S}_{\mathbb{I}\mathbb{H}_R}(c_2) & \text{if } c = c_1 \oplus c_2 \end{cases}$$

By observing the definition of $\mathcal{S}_{\mathbb{H}\mathbb{A}_R}$ and $[\iota_1, \iota_2]; \Psi$ (or, equivalently, $[\kappa_1, \kappa_2]; \Phi$), one can compute the value of $\mathcal{S}_{\mathbb{I}\mathbb{H}_R}$ on the basic operations of the signature $\Sigma_{\mathbb{H}\mathbb{A}_R}$ as follows:

$$\begin{aligned} \boxed{\bullet} &\mapsto [((1), \begin{pmatrix} 1 \\ 1 \end{pmatrix})] & \boxed{\circ} &\mapsto [(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, (1)), (\begin{pmatrix} 1 \\ 0 \end{pmatrix}, (1))] \\ \boxed{\bullet} &\mapsto [((1), ())] & \boxed{\circ} &\mapsto [((), (0))] & \boxed{k} &\mapsto [((1), (k))] \end{aligned}$$

The notation $[p_1, \dots, p_k]$ for an arrow in $\mathbb{S}\mathbb{V}_k[n, m]$ indicates the subspace of $n + m$ generated by p_1, \dots, p_k , where each p_i is a pair (\mathbf{x}, \mathbf{y}) of an n -vector $\mathbf{x} \in \mathbb{R}^n$ and an m -vector $\mathbf{y} \in \mathbb{R}^m$. The semantics of a basic operation $c \in \Sigma_{\mathbb{H}\mathbb{A}_R}^{op}$ is the subspace generated by $\{(x, y) \mid (y, x) \in \mathcal{S}_{\mathbb{I}\mathbb{H}_R}(c^*)\}$, where c^* is a basic operation in $\Sigma_{\mathbb{H}\mathbb{A}_R}$ by definition of $(\cdot)^*$.

11 Instances of the Cube

In this section we briefly present some instances of the cube construction.

11.1 Interacting Hopf Algebras over a Field

In the case in which the PID under consideration is actually a field k , we can considerably simplify the equational presentation of $\mathbb{H}H_k$. This will be given as $\mathbb{H}A_k + \mathbb{H}A_k$ quotiented by the following equations:

$$\begin{array}{c}
 \boxed{k} = \boxed{k^{-1}} \quad (\text{Inv}) \\
 \\
 \begin{array}{ccc}
 \boxed{\text{counit} \circ \text{unit}} = \boxed{\text{counit} \circ \text{multiplication}} = \boxed{\text{comultiplication} \circ \text{unit}} & & \boxed{\text{counit} \circ \text{multiplication}} = \boxed{\text{counit} \circ \text{multiplication}} = \boxed{\text{comultiplication} \circ \text{unit}} \\
 \boxed{\text{counit} \circ \text{multiplication}} = \boxed{\text{counit} \circ \text{multiplication}} & & \boxed{\text{counit} \circ \text{multiplication}} = \boxed{\text{counit} \circ \text{multiplication}} \\
 \boxed{\text{counit} \circ \text{multiplication}} = id_0 & & \boxed{\text{counit} \circ \text{multiplication}} = id_0
 \end{array}
 \end{array}$$

Indeed, all the axioms of Definition 9 involving scalar circuits can be derived by using (Inv) and the axioms of $\mathbb{H}A_k$ and $\mathbb{H}A_k^{op}$. Also observe that (Inv) is a valid equation of $\text{Span}(\mathbb{H}A_k)$ as it corresponds to a pullback square in $\mathbb{H}A_k$. The semantics of $\mathbb{H}H_k$ is given by $\mathbb{S}V_k$, as the field of fractions over a field is the field itself.

11.2 Integers and Hopf Algebras

If the PID under consideration is the ring of integers \mathbb{Z} , then each circuit of $\mathbb{H}A_{\mathbb{Z}}$ is provided with an equivalent presentation not involving any scalar circuit \boxed{k} (with the exception of the antipode $\boxed{\text{antipode}}$). This is given by application of (A1), (A17) and (A18) — for instance:

$$\boxed{3} \stackrel{(A18)}{=} \boxed{\text{multiplication} \circ \text{multiplication} \circ \text{multiplication}} \stackrel{(A1)}{=} \boxed{\text{multiplication} \circ \text{multiplication} \circ \text{multiplication}} \quad (10)$$

This suggests that the theory $\mathbb{H}A_{\mathbb{Z}}$ can be actually presented by equations not involving any scalar circuit at all. To this aim, let us freely construct the theory $\mathbb{H}A$ of (commutative/cocommutative) Hopf Algebras from the signature consisting of $\boxed{\text{counit}}$, $\boxed{\text{multiplication}}$, $\boxed{\text{comultiplication}}$, $\boxed{\text{unit}}$, an antipode $\boxed{\text{antipode}}$, and the following equations:

$$\begin{array}{ccc}
 \boxed{\text{counit} \circ \text{unit}} = \boxed{\text{counit} \circ \text{unit}} & \boxed{\text{counit} \circ \text{multiplication}} = \boxed{\text{counit} \circ \text{multiplication}} & \boxed{\text{counit} \circ \text{multiplication}} = \boxed{\text{counit} \circ \text{multiplication}} \\
 \boxed{\text{counit} \circ \text{multiplication}} = \boxed{\text{counit} \circ \text{multiplication}} & \boxed{\text{counit} \circ \text{multiplication}} = \boxed{\text{counit} \circ \text{multiplication}} & \boxed{\text{counit} \circ \text{multiplication}} = \boxed{\text{counit} \circ \text{multiplication}} \\
 \boxed{\text{counit} \circ \text{multiplication}} = \boxed{\text{counit} \circ \text{multiplication}} & \boxed{\text{counit} \circ \text{multiplication}} = \boxed{\text{counit} \circ \text{multiplication}} & \boxed{\text{counit} \circ \text{multiplication}} = \boxed{\text{counit} \circ \text{multiplication}}
 \end{array}$$

$$\begin{array}{ccc}
\boxed{\text{triangle right} \bullet} = \boxed{\bullet \text{triangle left}} & & \boxed{\text{triangle right}} = \boxed{\bullet} \\
\boxed{\text{triangle left} \bullet} = \boxed{\bullet \text{triangle right}} & & \boxed{\text{triangle left}} = \boxed{\bullet} \\
\boxed{\bullet \text{triangle right}} = \boxed{\bullet \text{triangle left}} & & \boxed{\bullet \text{triangle left}} = \boxed{\bullet \text{triangle right}} \\
\boxed{\bullet \text{triangle left}} = \boxed{\bullet \text{triangle right}} & & \boxed{\bullet \text{triangle left}} = id_0 \\
\boxed{\bullet \text{triangle right}} = \boxed{\bullet \text{triangle left}} & & \boxed{\bullet \text{triangle right}} = id_0 \\
\boxed{\text{triangle left} \text{triangle right}} = \boxed{\bullet \bullet}
\end{array}$$

Next we propose a theory \mathbb{IH} of interacting Hopf Algebras, given as $\mathbb{HA} + \mathbb{HA}^{op}$ quotiented by the following equations:

$$\begin{array}{ccc}
\boxed{\text{triangle left} \text{triangle right}} = \boxed{} = \boxed{\text{triangle right} \text{triangle left}} & & (11) \\
\boxed{\text{triangle left} \bullet} = \boxed{\bullet \text{triangle right}} = \boxed{\text{triangle left} \text{triangle right}} & & \boxed{\text{triangle right} \bullet} = \boxed{\bullet \text{triangle left}} = \boxed{\text{triangle right} \text{triangle left}} \\
\boxed{\bullet \text{triangle left}} = \boxed{\bullet \text{triangle right}} & & \boxed{\bullet \text{triangle left} \bullet} = \boxed{\bullet \text{triangle right} \bullet} \\
\boxed{\bullet \text{triangle left} \bullet} = \boxed{\bullet \text{triangle right} \bullet} & & \boxed{\bullet \text{triangle left} \text{triangle right}} = \boxed{\bullet \text{triangle right} \text{triangle left}} \\
\boxed{\bullet \text{triangle left} \text{triangle right}} = \boxed{\bullet \text{triangle right} \text{triangle left}} & & \boxed{\bullet \text{triangle left} \bullet} = id_0 \\
\boxed{\bullet \text{triangle right} \bullet} = id_0 & & \boxed{\bullet \text{triangle right} \bullet} = id_0
\end{array}$$

We remark that $\boxed{\text{triangle left} \text{triangle right}} = \boxed{}$ in \mathbb{HA} and thus $\boxed{\text{triangle left}} = \boxed{\text{triangle right}}$ is derivable by using (11). It is well-known that \mathbb{Z} -matrices freely characterize Hopf Algebras, that is, $\mathbf{Mat} \mathbb{Z} \cong \mathbb{HA}$ and thus also $\mathbb{HA}_{\mathbb{Z}} \cong \mathbb{HA}$. Our conjecture is that $\mathbb{IH}_{\mathbb{Z}} \cong \mathbb{IH}$, meaning that the theory of interaction for $\mathbb{IH}_{\mathbb{Z}}$ can be presented by a *finite* set of operations and equations, namely the ones of \mathbb{IH} . This claim can be verified by defining an interpretation of circuits of $\mathbb{IH}_{\mathbb{Z}}$ as circuits of \mathbb{IH} , following the same pattern of (10). Clearly, axioms of \mathbb{IH} are reflected in $\mathbb{IH}_{\mathbb{Z}}$. Then, it suffices to show that, modulo such interpretation, one is able to derive in \mathbb{IH} the axioms presenting $\mathbb{IH}_{\mathbb{Z}}$. By the result of Section 10, this would yield a finite axiomatization \mathbb{IH} for \mathbb{Q} -subspaces, where \mathbb{Q} is the field of fractions on \mathbb{Z} , that is, the field of rational numbers.

11.3 The Theory of Stateful Connectors

We now consider the case in which our PID is the polynomial ring $\mathbb{Z}_2[X]$ of the field \mathbb{Z}_2 over one indeterminant X . Analogously to the case of \mathbb{Z} , we observe that $\mathbb{HA}_{\mathbb{Z}_2[X]}$ can be presented by finitely many operations and equations. For this purpose, let $\Sigma_{\mathbb{D}}$ be the signature consisting of $\boxed{\bullet}$, $\boxed{\bullet \bullet}$, $\boxed{\bullet \text{triangle left}}$, $\boxed{\bullet \text{triangle right}}$ and the operation $\boxed{\text{triangle left} \text{triangle right}}$, which we call the *delay* circuit. We define \mathbb{D} as the PROP freely generated by the signature $\Sigma_{\mathbb{D}}$ and the following equations.

$$\begin{array}{ccc}
\boxed{\bullet \text{triangle left}} = \boxed{} & \boxed{\bullet \text{triangle right}} = \boxed{} & \boxed{\text{triangle left} \text{triangle right}} = \boxed{\text{triangle left} \text{triangle right}} \\
\boxed{\bullet \bullet} = \boxed{} & \boxed{\bullet \bullet} = \boxed{\bullet \bullet} & \boxed{\bullet \text{triangle left} \bullet} = \boxed{\bullet \text{triangle right} \bullet} \\
\boxed{\bullet \text{triangle left} \bullet} = \boxed{\bullet \text{triangle right} \bullet} & \boxed{\bullet \text{triangle left} \text{triangle right}} = \boxed{\bullet \text{triangle right} \text{triangle left}} & \boxed{\bullet \text{triangle left} \text{triangle right}} = \boxed{\bullet \text{triangle right} \text{triangle left}}
\end{array}$$

$$\begin{array}{ccc}
\boxed{\text{Delay}} = \boxed{\text{Store}} & & \boxed{\text{Store}} = \boxed{\text{Delay}} \\
\boxed{\text{Store}} = \boxed{\text{Store}} & & \boxed{\text{Store}} = \boxed{\text{Store}} \\
\boxed{\text{Store}} = \boxed{\text{Store}} & & \boxed{\text{Store}} = id_0 \\
\boxed{\text{Store}} = \boxed{\text{Store}} & & \boxed{\text{Store}} = id_0
\end{array}
\tag{ASep}$$

Observe that the *anti-separability* equation (ASep) is an instance of (A18) for $k_1 = 1 \in \mathbb{Z}_2[X]$ and $k_2 = -1 = 1 \in \mathbb{Z}_2[X]$. One can think of \mathbb{D} as the theory of anti-separable (commutative/cocommutative) bialgebras with a delay operation commuting with the bialgebraic structure. In order to verify that \mathbb{D} is isomorphic to $\mathbb{H}\mathbb{A}_{\mathbb{Z}_2[X]}$, we define an interpretation of circuits of $\mathbb{H}\mathbb{A}_{\mathbb{Z}_2[X]}$ as circuits of \mathbb{D} by mapping \boxed{X} into $\boxed{\text{Delay}}$ and any other scalar circuit \boxed{k} into its decomposition in terms of $\boxed{\text{Delay}}$ and $\boxed{\text{Store}}$ given by (A18), (A1) and (A17). For instance:

$$\boxed{X^2+1} \stackrel{(A18)}{=} \boxed{\text{Circuit}} \text{ is interpreted as } \boxed{\text{Circuit}}.$$

This interpretation clearly reflects the axioms of \mathbb{D} and thus it suffices to show that it preserves the axioms of $\mathbb{H}\mathbb{A}_{\mathbb{Z}_2[X]}$, which can be verified by writing down the corresponding derivations in \mathbb{D} .

Similarly to the case of \mathbb{Z} , we conjecture that the theory $\mathbb{H}\mathbb{H}_{\mathbb{Z}_2[X]}$ can be finitely presented as $\mathbb{D} + \mathbb{D}^{op}$ quotiented by the following equations.

$$\begin{array}{ccc}
\boxed{\text{Delay}} = \boxed{\text{Store}} = \boxed{\text{Delay}} & & \boxed{\text{Store}} = \boxed{\text{Store}} \\
\boxed{\text{Store}} = \boxed{\text{Store}} = \boxed{\text{Store}} & & \boxed{\text{Store}} = \boxed{\text{Store}} = \boxed{\text{Store}} \\
\boxed{\text{Store}} = \boxed{\text{Store}} & & \boxed{\text{Store}} = \boxed{\text{Store}} \\
\boxed{\text{Store}} = \boxed{\text{Store}} & & \boxed{\text{Store}} = \boxed{\text{Store}} \\
\boxed{\text{Store}} = \boxed{\text{Store}} & & \boxed{\text{Store}} = \boxed{\text{Store}} \\
\boxed{\text{Store}} = id_0 & & \boxed{\text{Store}} = id_0
\end{array}$$

We conclude by briefly describing the operational intuition motivating the theory studied in this section. In [1], we explained how $\mathbb{H}\mathbb{H}_{\mathbb{Z}_2}$ corresponds to the algebra of *stateless* connectors [2]. By adding the unknown X , we obtain an algebra of *stateful* connectors: the circuit $\boxed{\text{Delay}}$ acts as a *delay* that stores the value arriving on the left and outputs on the right the value kept by the store. This intuition has a neat formal understanding in terms of *streams*: rational streams on \mathbb{Z}_2 are in one-to-one correspondence with rationals in $\mathbb{Z}_2[X]$ (see e.g. [13]). This fact holds more generally for any field k . Since the polynomial ring $k[X]$ is always a PID, we can apply the same constructions of this paper in order to obtain a theory of circuits characterizing subspaces of streams over k . An interesting application of this is, for instance, a sound and complete axiomatization for *signal flow graphs*.

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A The Frobenius Laws in \mathbb{HH}_R^w

The Frobenius axioms both for the white — (S2) — and for the black structure — (S3) — make valid any deformation of the internal topology of circuits of \mathbb{HH}_R^w , as long as the connections between boundaries are preserved. We list here some useful laws of that kind. In describing the various derivation steps, we occasionally use the notation $(n)^{op}$, which means the counterpart in \mathbb{HA}_R^{op} of a valid equation (n) in \mathbb{HA}_R .

$$\begin{array}{c} \text{Diagram 1} \end{array} \stackrel{(S3)}{=} \begin{array}{c} \text{Diagram 2} \end{array} \stackrel{(A7),(A6)}{=} \begin{array}{c} \text{Diagram 3} \end{array} \stackrel{(A6)}{=} \begin{array}{c} \text{Diagram 4} \end{array} \stackrel{(S3)}{=} \begin{array}{c} \text{Diagram 5} \end{array} \quad (12)$$

$$\begin{array}{c} \text{Diagram 6} \end{array} \stackrel{(S3)}{=} \begin{array}{c} \text{Diagram 7} \end{array} \stackrel{(A7),(A6),(A6)^{op}}{=} \begin{array}{c} \text{Diagram 8} \end{array} \stackrel{(A6),(A7)^{op},(A6)^{op}}{=} \begin{array}{c} \text{Diagram 9} \end{array} \stackrel{(S3)}{=} \begin{array}{c} \text{Diagram 10} \end{array} \quad (13)$$

The following laws are derived analogously. The ones involving the white structure use the white Frobenius axiom (S2).

$$\begin{array}{c} \text{Diagram 11} \end{array} = \begin{array}{c} \text{Diagram 12} \end{array} = \begin{array}{c} \text{Diagram 13} \end{array} \quad (14) \qquad \begin{array}{c} \text{Diagram 14} \end{array} = \begin{array}{c} \text{Diagram 15} \end{array} = \begin{array}{c} \text{Diagram 16} \end{array} \quad (15)$$

$$\begin{array}{c} \text{Diagram 17} \end{array} = \begin{array}{c} \text{Diagram 18} \end{array} = \begin{array}{c} \text{Diagram 19} \end{array} \quad (16) \qquad \begin{array}{c} \text{Diagram 20} \end{array} = \begin{array}{c} \text{Diagram 21} \end{array} = \begin{array}{c} \text{Diagram 22} \end{array} \quad (17)$$

For later reference, we also record the following derivation.

$$\begin{array}{c} \text{Diagram 23} \end{array} \stackrel{(A10)}{=} \begin{array}{c} \text{Diagram 24} \end{array} \stackrel{(A9)^{op}}{=} \begin{array}{c} \text{Diagram 25} \end{array} \stackrel{(A2)}{=} \begin{array}{c} \text{Diagram 26} \end{array} \quad (18)$$

The same equation “reflected about the y -axis” and the black counterparts are proven analogously.

$$\begin{array}{c} \text{Diagram 27} \end{array} = \begin{array}{c} \text{Diagram 28} \end{array} \quad (19) \qquad \begin{array}{c} \text{Diagram 29} \end{array} = \begin{array}{c} \text{Diagram 30} \end{array} \quad (20) \qquad \begin{array}{c} \text{Diagram 31} \end{array} = \begin{array}{c} \text{Diagram 32} \end{array} \quad (21)$$

B Shaping the Compact Closed Structure of \mathbb{HH}_R^w

In this section we give more detailed proofs to the statements of Section 5.

Proof (Proof of Proposition 5). In order to show (CC1), we proceed by induction on n . For the case $n = 1$, the statement is given by (13). For the inductive step, let $n = i + 1$. In the sequel we show the equality

$$\begin{array}{c} \text{Diagram 33} \end{array} = \begin{array}{c} \text{Diagram 34} \end{array} \quad (22)$$

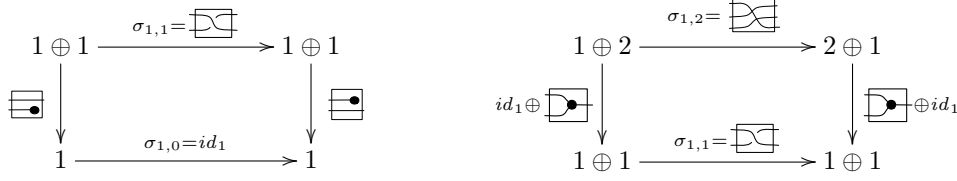
yielding the left side of (CC1). The right side has a completely analogous derivation. For this purpose, it will be useful the following equation, allowing to “move” the compact closed structure past the symmetries of \mathbb{HH}_R^w .

$$\begin{array}{c} \text{Diagram 35} \end{array} = \begin{array}{c} \text{Diagram 36} \end{array} \quad (23)$$

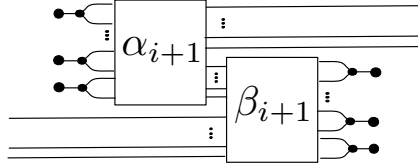
Its derivation in \mathbb{HH}_R^w is the following.

$$\begin{array}{c} \text{Diagram 37} \end{array} = \begin{array}{c} \text{Diagram 38} \end{array} = \begin{array}{c} \text{Diagram 39} \end{array} = \begin{array}{c} \text{Diagram 40} \end{array} = \begin{array}{c} \text{Diagram 41} \end{array}$$

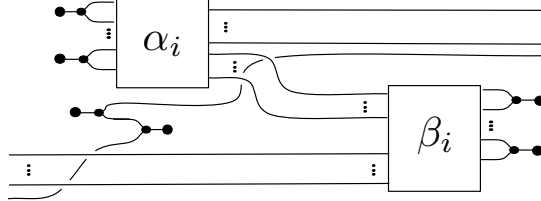
The first and the second equality holds by naturality of symmetry, applied as on the left and on the right below, respectively.



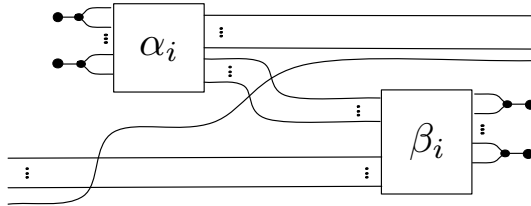
The third equality applies the axiom $\sigma_{1,2} = (\sigma_{1,1} \oplus id_1); (id_1 \oplus \sigma_{1,1})$ of symmetric monoidal categories (SMCs). Finally, the fourth equality applies the axiom $\sigma_{1,1}; \sigma_{1,1} = id_1$ of SMCs. We are now ready to show the derivation of (22). The circuit on the left side of (22) has the following shape.



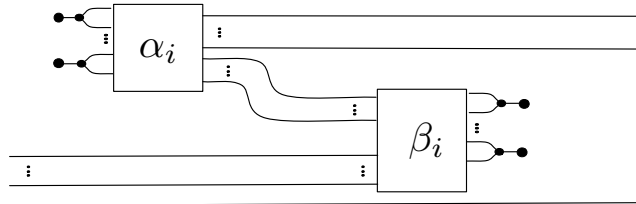
By definition, port 1 of the bottommost circuit $\boxed{\bullet \rightarrow \bullet}$ (call it c_l) connects to the bottommost port of the right boundary and port 2 connects to port 1 of the bottommost circuit $\boxed{\bullet \leftarrow \bullet}$ (call it c_r). The other port of c_r connects instead to the bottommost port on the left boundary. By iteratively applying (23) to c_r , we can move it towards the middle of the circuit, past all the symmetries in β_{i+1} . The resulting circuit is the following:



Note that, now that we isolated c_l and c_r , the circuits α_{i+1} and β_{i+1} become by definition α_i and β_i — observe that the application of (23) does not affect the arity of the symmetries in the circuit. We are now in position to apply (13):



We can then use again (23) to move the identity circuit in the middle towards the bottom.

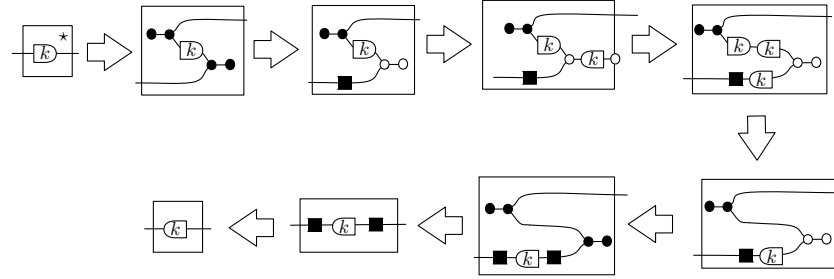


It is now possible to apply the inductive hypothesis on i , obtaining as a result the desired identity circuit as on the right side of (22). \square

Proof (Proof of Proposition 6). The proof is by induction on $c \in \mathbb{IH}_R^w$. First we give the derivations for the four base cases of white/black unit/counit.

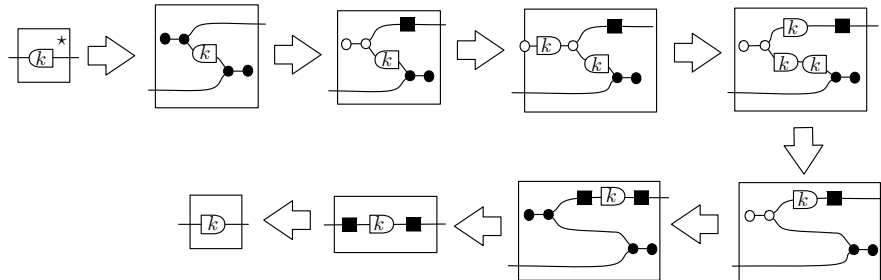
$$\begin{array}{c}
\boxed{\begin{array}{c} \star \\ \bullet \end{array}} \stackrel{\text{Def. } (\cdot)^*}{=} \boxed{\begin{array}{c} \bullet \end{array}} \stackrel{(A6)^{op}}{=} \boxed{\bullet} \\
\\
\boxed{\begin{array}{c} \star \\ \bullet \end{array}} \stackrel{\text{Def. } (\cdot)^*}{=} \boxed{\begin{array}{c} \bullet \end{array}} \stackrel{(A6)}{=} \boxed{\bullet} \\
\\
\boxed{\begin{array}{c} \star \\ \circ \end{array}} \stackrel{\text{Def. } (\cdot)^*}{=} \boxed{\begin{array}{c} \circ \end{array}} \stackrel{(D2), (19)}{=} \boxed{\begin{array}{c} \blacksquare \end{array}} \stackrel{(A3)}{=} \boxed{\begin{array}{c} \blacksquare \end{array}} \stackrel{(A10)}{=} \boxed{\circ} \\
\\
\boxed{\begin{array}{c} \star \\ \circ \end{array}} \stackrel{\text{Def. } (\cdot)^*}{=} \boxed{\begin{array}{c} \bullet \end{array}} \stackrel{(D1)}{=} \boxed{\begin{array}{c} \circ \end{array}} \stackrel{(A10)^{op}}{=} \boxed{\begin{array}{c} \circ \end{array}} \stackrel{(A3)^{op}}{=} \boxed{\circ}
\end{array}$$

We now consider the base case \boxed{k} , for $k \in R$.



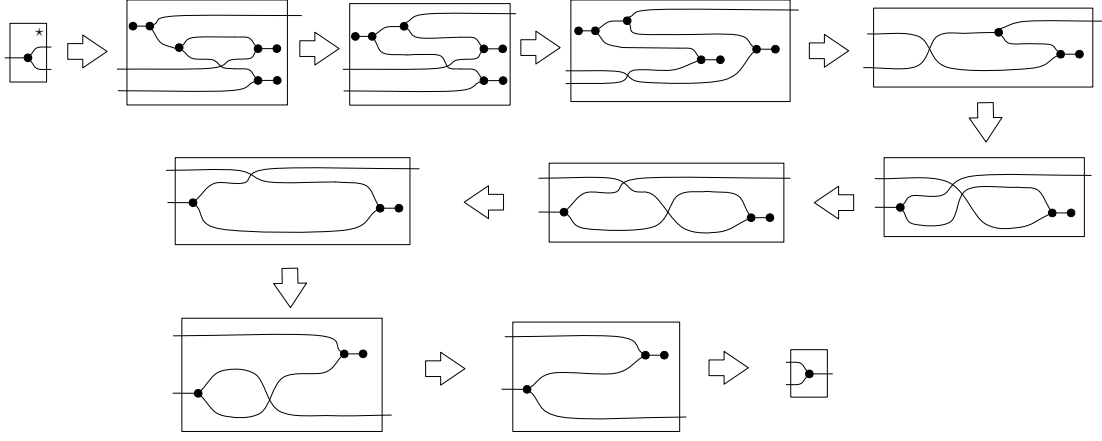
The first step of the derivation is simply unfolding the definition of $(\cdot)^*$. The following laws are then applied in the derivation above: (D2), $(A10)^{op}$, (S9), (S1), (S5), (13), (A2).

Next we give the derivation for the base case \boxed{k} , for $k \in R$.



The following laws are applied in sequence: definition of $(\cdot)^*$, (D1) and (18), $(A10)$, (S8), (S1), (S4) and (20), (13), $(A2)^{op}$.

Next we provide the derivation for the base case $\boxed{\bullet}$.



The sequence of applied laws is: definition of $(\cdot)^*$, (A8), (23), (13), naturality of symmetry, axiom of SMCs, (A7)^{op}, (23), (A7), (14).

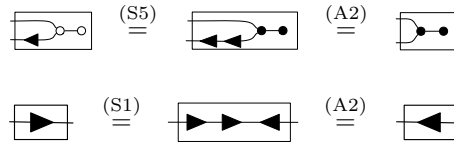
The remaining base cases of operations $\boxed{\bullet \circ}$, $\boxed{\circ \bullet}$ and $\boxed{\bullet \bullet}$ are handled in an analogous way by using the Frobenius laws derived in Appendix A. The proof is concluded by examining the two inductive cases. For sequential composition:

$$\left(\begin{array}{c} n \\ \hline \end{array} \boxed{c_1} \begin{array}{c} \text{---} z \text{---} \\ \hline \end{array} \boxed{c_2} \begin{array}{c} m \\ \hline \end{array} \right)^* \stackrel{\text{Funct. } (\cdot)^*}{=} \begin{array}{c} m \\ \hline \end{array} \boxed{c_2^*} \begin{array}{c} \text{---} z \text{---} \\ \hline \end{array} \boxed{c_1^*} \begin{array}{c} n \\ \hline \end{array} \stackrel{\text{IH}}{=} \begin{array}{c} m \\ \hline \end{array} \boxed{c_2^R} \begin{array}{c} \text{---} z \text{---} \\ \hline \end{array} \boxed{c_1^R} \begin{array}{c} n \\ \hline \end{array} \stackrel{\text{Def. } (\cdot)^R}{=} \left(\begin{array}{c} n \\ \hline \end{array} \boxed{c_1} \begin{array}{c} \text{---} z \text{---} \\ \hline \end{array} \boxed{c_2} \begin{array}{c} m \\ \hline \end{array} \right)^R$$

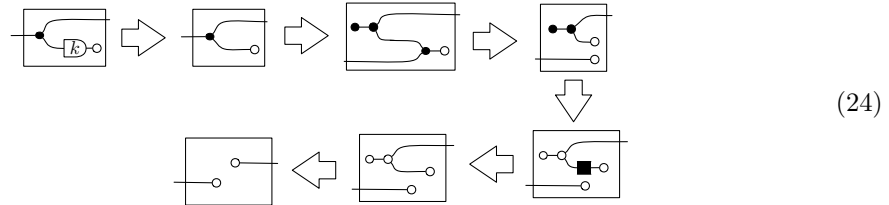
The derivation for the case of parallel composition \oplus is analogous. \square

C Derived Laws of \mathbb{H}_R^w

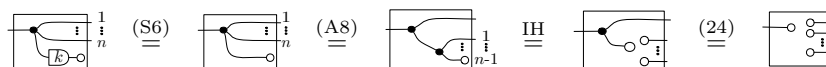
In this section we supply the equational proofs of the laws stated in Section 4. We begin with the derivations of (D2) and (D3).



The derivation of (D1) is analogous to the one of (D2), with (S4) used in place of (S5). In order to show the validity of (D4), we proceed by induction on the coarity $n \geq 1$ of the circuit, i.e., the number of gates on the right boundary. For the case $n = 1$, we have the following derivation.



The sequence of applied laws is: (S6), (12), (A14)^{op}, (D2), (A10), (A3)^{op}. The inductive case is handled as follows.



Next we show the derivation of (D5). The one of (D6) is analogous.

$$\begin{array}{c}
 \boxed{\text{diagram}} \stackrel{(12)}{=} \boxed{\text{diagram}} \stackrel{(A11)^{op}}{=} \boxed{\text{diagram}} \stackrel{(CC3), \text{Prop. 6}}{=} \boxed{\text{diagram}} \stackrel{(A2)}{=} \boxed{\text{diagram}} \stackrel{(12)}{=} \boxed{\text{diagram}}
 \end{array}$$

We now consider the task of deriving law (D7). First, it is useful to record the following derivation.

$$\begin{array}{c}
 \boxed{\text{diagram}} \Rightarrow \boxed{\text{diagram}} \Rightarrow \boxed{\text{diagram}} \Downarrow \\
 \boxed{\text{diagram}} \Leftarrow \boxed{\text{diagram}} \Leftarrow \boxed{\text{diagram}} \Leftarrow \boxed{\text{diagram}}
 \end{array} \tag{25}$$

The first step uses twice (13). The second step is valid by Proposition 6. The successive step uses in sequence: (14), (D2) and (18), (16), (A2). We now ready to derive the first half of (D7).

$$\begin{array}{c}
 \boxed{\text{diagram}} \Rightarrow \boxed{\text{diagram}} \Rightarrow \boxed{\text{diagram}} \Downarrow \\
 \boxed{\text{diagram}} \Leftarrow \boxed{\text{diagram}} \Leftarrow \boxed{\text{diagram}}
 \end{array}$$

The sequence of equations that are used is the following: (25), sliding (naturality of \oplus), (A8) and (A5), (A18) and (A17), (A6) and (A3). The second half of (D7) is derived analogously as follows.

$$\begin{array}{c}
 \boxed{\text{diagram}} \Rightarrow \boxed{\text{diagram}} \Rightarrow \boxed{\text{diagram}} \Downarrow \\
 \boxed{\text{diagram}} \Leftarrow \boxed{\text{diagram}} \Leftarrow \boxed{\text{diagram}}
 \end{array}$$